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Tentamen i Kursen DN2221
Tillämpade Numeriska Metoder I
Thursday 2012-12-13 kl 14–19

SOLUTIONS

1. a) Columns of S are the eigenvectors, diagonal elements of D are the eigenvalues.

b) The eigenvalues of A are $-1-\alpha$ and $-1+\alpha$. The corresponding eigenvectors are $(1,-1)^T$ and $(1,1)^T$. The matrix is diagonalizable if the eigenvalues are different. When $\alpha = 0$ they are equal, but then the matrix is already diagonal.

2. a) When $a > 0$, the general solution is $u(t) = Acos(\sqrt{a}t) + Bsin(\sqrt{a}t)$, which is stable. When $a < 0$, the general solution is $u(t) = Aexp(\sqrt{-a}t) + Bexp(-\sqrt{-a}t)$, which is unstable.

b) The characteristic equation is $\lambda^2 + a = 0$. When $a > 0$, $h\lambda_i$ are situated on the imaginary axis (out side the stability area of the explicit Euler method and the numerical solution is unstable. When $a < 0$, $h\lambda_i$ is real, one posistive and one negative. The positive value is outside the stability region and the Euler solution is unstable.

c) For implicit Euler the numerical solution is stable when $a > 0$ When $a < 0$ the solution is unstable when $0 < h\sqrt{a} < 2$ and stable when $h\sqrt{a} \geq 2$.

3. See, the book, appendix A.3. The solution is $a = 1, b = -2, c = 1$ and the order is $p = 1$.

4. The roots of the characteristic equation are $\lambda = 50 \pm \sqrt{50^2 + 21}$, hence one root is close to 100, the other root is a negative number ≈ -0.21, The general solution is $u(x) \approx Aexp(100x) + Bexp(-0.2x)$ and (very) unstable. If $A = 1$ the solution at $x = 1$ is $\approx 10^{43}$. Solving this problem as an IVP numerically will give unstable solutions (unless $A = 0$) both for a stiff and nonstiff method. Solving as a BVP will give oscillations unless h is small enough, but this can be handled with the FDM.

5. a) The matrix A is diagonalizable with real eigenvalues for all $alpha$-values (see problem 1), hence the system is hyperbolic.

b) The eigenvalue problem $AS = AD$ gives $S^{-1}AS = D$, where S and D are given in problem 1. Make the transformation $u = Sv$ and we get the system $v_t + S^{-1}ASv = 0$, in component form $v_{1t} + (-1-\alpha)v_{1x} = 0$ and $v_{2t} + (-1+\alpha)v_{2x} = 0$.

c) Inserting $\alpha = 3$ gives the characteristics $t = -4x + C$ and $t = 2x + C$.

6. a) Parabolic

b) Discretize the r-axis according to $r_0 = -h_r, r_1 = a, r_2 = a + h_r, r_i = (i-1)h, r_N = R, r_{N+1} = R + h_r$, i.e. $h_r = (R-a)/(N-1)$. Use the MoL to obtain a system of N ODEs at $r = r_1, r_2, \ldots r_N$ with z as independent variable. The discretized boundary conditions are $(c_2 - c_0)/2h = 0$ and $(c_{N+1} - c_{N-1})/2h = 0$ After elimination of c_0 and c_{N+1} we get the following system of ODEs

$$at \quad r = r_1 = 0: \qquad \frac{dc_1}{dz} = 2\frac{D}{v}\frac{c_2 - c_1}{h^2}$$

$$at \quad r = r_i: \qquad \frac{dc_i}{dz} = \frac{D}{v}(\frac{c_{i+1} - 2c_i + c_{i-1}}{h_r^2}) + \frac{1}{r_i}\frac{c_{i+1} - c_{i-1}}{2h_r}), i = 2, 3, \ldots, N-1$$

$$at \quad r = r_N: \qquad \frac{dc_N}{dz} = 2\frac{D}{v}\frac{-c_N + c_{N-1}}{h_r^2}$$

and written in vector form

$$\frac{d\mathbf{c}}{dz} = B\mathbf{c}, \quad \mathbf{c}(0) = \mathbf{c}_0$$

with B triangular.

c) $T = I - h_t B$, $M = I$ and $g = 0$.