Homework 2, DN2230

(Updated version)
Due November 29, 2012

If a correct solution is handed in before the deadline two bonus points will be awarded to the final written exam. If a solution that is handed in before that date is not correct, it has to be redone, but the second time without yielding bonus points for the exam.

Solutions must be clearly written, and easy to follow. If not, they will not generate bonus points, and must be redone.

A written solution (hard-copy) should be handed in during the lecture or in the “homework mailbox” at the mathematics studentexpedition on Lindstedsvägen 25. Also, please send me your code by e-mail.

1. Implement the full multigrid algorithm. You may do this by extending your multigrid program from homework 1. By section 3.2.2 in “Multigrid”, by Trottenberg et al. it should be possible to choose the ingredients in the algorithm such that the FMG error

   $$\|u_h^{FMG} - u_h\|$$

   is of the same (or higher) order in the discretization step length h as the discretization error

   $$\|u - u_h\|.$$

   Here u is the PDE solution, e.g. the solution to Poisson's equation in one dimension, $-u'' = f$ in $(0, 1)$, u_h is the exact solution to the discretized problem with step length h, and u_h^{FMG} is the corresponding full multigrid solution.

   Verify numerically that you can choose the ingredients in the FMG algorithm (interpolation, MG iterations per level etc.) such that the FMG error is of the same (or higher) order in h as the discretization error.

2. Adapt the Power Iteration method so that it is possible to find the second largest (in magnitude) eigenvalue of a symmetric matrix, and the corresponding eigenvector. Give a Matlab program that uses your

algorithm to compute these for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 9 \end{pmatrix}.$$

3. Carry out the Rayleigh quotient iteration on the matrix in exercise 2 with starting vector $x_0^T = (1, 1, 1)$. Illustrate the convergence by plotting the eigenvalue approximation error as a function of iteratation (in a semilogy plot) and explain what you expect from theory. Change $a_{1,3}$ to 4, carry out the Rayleigh quotient again and explain what you expect from theory and observe in the error plot.

4. (This exercise has been replaced by exercise 6-7.) ~~Exercise 7.3 in "Numerical Linear Algebra" (NLA).~~

5. (This exercise has been replaced by exercise 6-7.) ~~What is the computational work in flops to perform a QR factorization of a symmetric tridiagonal matrix $A \in \mathbb{R}^{m\times m}$? We are primarily interested in the order of the method; hence, the task here is to determine an optimal number α, such that the computational work is less than or equal to a constant times m^α.~~

6. Implement Algorithm 28.1 in Trefethen&Bau (QR-method without shifts) and apply it to the matrix generated by

```
nn=4;
A=full(gallery('wathen',nn,nn));
m=length(A);
```

   Visualize the error for the first 1000 iterations as a function of iteration by plotting the norm of the elements below the diagonal using `norm(tril(A,-1),1)` as well as the difference between the diagonal elements and the eigenvalues computed with `eig`. In this exercise you may use the matlab command `qr`.

7. Implement the two-phase approach for the eigenvalue problem, described in Trefethen&Bau Chapter 25. That is, first reduce the matrix to Hessenberg form (or tridiagonal form) using Algorithm 26.1 and subsequently carry out the QR-method (without shifts implemented in exercise 6). Do timing comparison with the commands `tic` and `toc` for the matrix in exercise 6 for different values of n. Plot the timing comparison for the two phases. Relate the timing with the computational complexity expected from theory.

Good luck!