Exercise questions to prepare for the written exam
Part 2/2
DN2230

41. Prove the convergence of the Arnoldi method for eigenvalue problems (under appropriate assumptions) by bounding

$$\|(I - Q_k Q_k^T)x_i\|,$$

where x_i is an eigenvector and Q_k is the basis matrix generated by the Arnoldi method.

Hint: This is not included in the course book TB. It was covered in the lectures. The main result can be found in the online version of Saad, Numerical methods for large eigenvalue problems `http://www-users.cs.umn.edu/~saad/eig_book_2ndEd.pdf`, in particular Lemma 6.2 and Proposition 6.10.

42. The approximations generated by GMRES are said to be equivalent to the linear least squares problem

$$\min_{z\in\mathbb{R}^n} \|AK_n z - b\|_2, \tag{1}$$

where $K_n = (b, Ab, \ldots, A^{n-1}b)$. Describe the equivalence. GMRES better than solving the least squares problem (1) directly. Describe two reasons why GMRES is better.

43. What is the difference between the largest eigenvalue and the extreme eigenvalue? Which eigenvalue is normally found with the power method? Which eigenvalue is normally favored in the Arnoldi method?

44. The Arnoldi method can be seen as a procedure to compute an Arnoldi relation (sometimes called the Arnoldi factorization). Give the definition of an Arnoldi relation. Suppose we run the Arnoldi method for a matrix $A \in \mathbb{R}^{m\times m}$ and at iteration $k < m$ find that the Hessenberg matrix is not unreduced, i.e., $h_{k+1,k} = 0$. Show that the eigenvalues generated by the Arnoldi method for eigenvalue problems in this case are eigenvalues of A (if we disregard rounding error).

45. Suppose $QR = A \in \mathbb{R}^{2\times 2}$ is a QR-decomposition of A. State three other QR-decompositions of A.

46. State the optimality property of the Conjugate Gradient in the norm $\|\cdot\|_{A^{-1}}$. Derive a corresponding optimality optimality property in the norm $\|\cdot\|_A$.

47. Consider the following formula (called the Sherman-Morrison-Woodbury formula). If $B = A + uv^T$, then

$$B^{-1}w = A^{-1}w - \frac{v^T A^{-1} w}{1 + v^T A^{-1} u} A^{-1}u.$$

Suppose $A \in \mathbb{R}^{m\times m}$ is a sparse matrix and $(A - \mu I)^{-1}z$ can be computed in $O(m)$ and $(B - \mu I)^{-1}z$ can be computed in $O(m^2)$. Improve the shift-and-invert Arnoldi method for B such that, carrying out k steps has complexity $O(km)$ when $k \ll m$.

48. Describe the Gram-Schmidt procedure. Describe two procedures to remedy the numerical instability with the Gram-Schmidt procedure.

49. Give the definition of a well separated block and an (diagonal) adjacent block of a matrix. Given a matrix $A \in \mathbb{R}^{n\times n}$ and $n = 2^L$, describe a recursive subdivision procedure that generates large well separated blocks.

50. Let A be the $N \times N$ matrix with entries $a_{ij} = 1/|i-j|$ when $i \neq j$, and $a_{ij} = 0$ for $i = j$. Describe an algorithm for efficient computation of the matrix-vector product Ax (with some given vector x).

You may use that if a submatrix A_{lmk} of A is separated from the diagonal, then there exists a $k \times k$-matrix B_{lmk} with rank $J + 1$ such that

$$|(A_{lmk})_{ij} - (B_{lmk})_{ij}| \leq 4^{-J}.$$

Here A_{lmk} denotes a submatrix of A containing all elements a_{ij} of A such that $l \leq i < l + k$ and $m \leq j < m + k$. Moreover, B_{lmk} can be computed explicitly as

$$B_{lmk} = \bar{B}^T \bar{A} \bar{\Gamma},$$

where $\bar{B}$ and $\bar{\Gamma}$ are $(J+1) \times k$-matrices, and $\bar{A}$ is an $(J+1) \times (J+1)$-matrix.

What will be the computational complexity of your proposed algorithm?