DN2230 Fast Numerical Algorithms for Large-scale Problems
Exam

Aids: None **Time: Four hours**

Grades: E: 16 points, D: 19 points, C: 22 points, B: 25 points, A: 28 points (out of the possible 35 points, including bonus points from homeworks).

Problem 1 (4p)

a. What is the Rayleigh quotient iteration?

b. Describe the convergence of the Rayleigh quotient iteration (for symmetric as well as unsymmetric matrices).

Problem 2 (4p) Show that

$$\|r_n\| = \min_{p_n \in P_n} \|p_n(A)b\|,$$

where r_n is the residual in step n of the GMRES method, and

$$P_n = \{\text{polynomials } p \text{ of degree} \leq n \text{ with } p(0) = 1\}.$$

Problem 3 (2p) Give an example of a prolongation operator.

Problem 4 (5p)

a. Describe the damped Jacobi method.

b. Show that the fixed points of damped Jacobi method are the solutions to the corresponding linear systems of equations (under appropriate assumptions on A).

c. Derive a matrix R_ω such that the error in the damped Jacobi method satisfies

$$e_k = R_\omega e_{k-1}.$$

Problem 5 (5p) From the definition of the Conjugate Gradient method it can be shown that $x_n \in \mathcal{K}_n$, and that the residuals satisfy $r_n \perp \mathcal{K}_n$. Use this to show that x_n is the unique point in $\mathcal{K}_n$ that minimizes $\|e_n\|_A$. (Recall also that we need A to be symmetric positive definite in order to be able to use the CG method.)

Problem 6 (4p) Consider the Arnoldi method for eigenvalue problems applied to the matrix $A \in \mathbb{R}^{m\times m}$ with starting vector $\alpha_1 x_1 + \cdots + \alpha_m x_m$, where $x_1, \ldots, x_m$ are eigenvectors of A. Prove the error bound

$$\|(I - Q_k Q_k^T)x_i\|_2 \le \varepsilon_i^{(k)} \xi_i,$$

where Q_k is the basis matrix generated by k steps of the Arnoldi method and

$$\xi_i = \sum_{j=1, j\neq i}^{m} \frac{|\alpha_j|}{|\alpha_i|}$$

and

$$\varepsilon_i^{(k)} = \min_{p\in\Pi_{k-1}, p(\lambda_i)=1} \max_{j\neq i} |p(\lambda_j)|$$

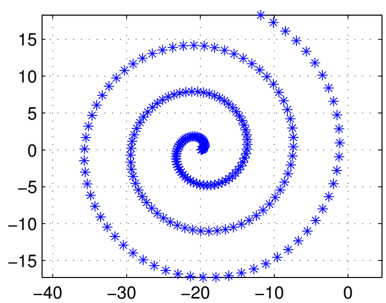
where $\Pi_k = \{\text{polynomials } p \text{ of degree} \le k\}$.

Problem 7 (5p)

Consider a matrix $A \in \mathbb{R}^{n\times n}$ with eigenvalues

$$\lambda_i = z_i e^{iz_i} - 20,$$

with $z_1 = 0.1, z_2 = 0.2, z_3 = 0.3, \ldots, z_n = 20$. The eigenvalues are also given in the figure to the right.



Consider the Arnoldi method applied to A and neglect rounding errors, and suppose the starting vector is chosen such that we do not have premature breakdown.

a. Use the eigenvector error bound in Problem 6 and show that we have

$$\varepsilon_{200}^{(k)} \le \left(\frac{20 - 0.1}{20}\right)^{k-1}.$$

b. What is the eigenvalue error corresponding to λ_{200} after 200 iterations?