Exercise questions to prepare for the written exam

1 Exercises

Exercises marked with * are considered somewhat more difficult.

Exercises about sparse eigenvalue problems

1. Formulate the power method and the Rayleigh quotient iteration.

2. Formulate the convergence of the power method and Rayleigh quotient iteration.

3. Suppose the Rayleigh quotient iteration for a matrix A is started with a vector v. Estimate the expected number of steps required to reach machine precision (error 10^{-16}) when

   (a) A is non-symmetric,

   (b) A is symmetric.

   Carry out the estimation by replacing $\|v_k - v_*\| = \mathcal{O}(\|v_{k-1} - v_*\|^p)$ by $\|v_k - v_*\| = \beta\|v_{k-1} - v_*\|^p$ where β can be considered constant.

4. Suppose $Av_* = \lambda_* v_*$ where $v_* \in \mathbb{C}^m$, $\lambda_* \in \mathbb{C}$ and let r be the Rayleigh quotient $r(v) = v^*Av/v^*v$.

   (a) Show that the Rayleigh quotient is the eigenvalue approximation associated with a given eigenvector approximation which minimizes the residual

   $$\min_{\alpha\in\mathbb{R}} \|Av - \alpha v\|_2 = \|Av - r(v)v\|_2.$$

   Hint: Use the normal equations solve the linear least squares problem with one unknown variable.

   (b) Suppose $v \in \mathbb{R}^m$ and $v_* \in \mathbb{R}^m$. Derive the vector $q_* \in \mathbb{R}^m$ such that the Taylor expansion of the Rayleigh quotient is

   $$r(v) = \lambda_* + q_*^T(v - v_*) + O(\|v - v_*\|^2).$$

   (c*) If $v_* \in \mathbb{C}^m\backslash\mathbb{R}^m$ is complex a different proof technique is necessary. (The Rayleigh quotient is not complex differentiable and does not have complex Taylor expansion, in particular v^*v is not analytic.) Prove the following properties for the complex case

   $$\begin{aligned} v^*v &= v_*^*v_* + 2\,\mathrm{Re}((v - v_*)^*v_*) + \|v - v_*\|_2^2 \\ v^*Av &= \lambda_* v_*^* v_* + \mathcal{O}(\|v - v_*\|) \end{aligned}$$

Use these properties and show that

$$r(v) = \lambda_* v_*^* v_* + \mathcal{O}(\|v - v_*\|)$$

5. What are the advantages and disadvantages of classical Gram-Schmidt, modified Gram-Schmidt and repeated Gram-Schmidt (double and triple)?

6. Let $w_k = A^k w_0 / \|A^k w_0\|$ be the iterates of the power method. Let $Q_n \in \mathbb{C}^{m\times n}$ and $\underline{H}_n$ be an Arnoldi factorization.

   (a) Show that all eigenvalues of $H_n = Q_n^* A Q_n$ equals all solutions to the generalized eigenvalue problem

   $$\mu W_n^T W_n z = H_n z \tag{1}$$

   where $W_n = (w_0, \ldots, w_{n-1})$.

   (b) Why is the Arnoldi method to prefer over the computation of the solutions to (1)?

7. Suppose the eigenvalues of a matrix A are as follows

   $$\begin{aligned} \lambda_1 &= 10 \\ \lambda_k &= \sin(k) + i\cos(k), \end{aligned}$$

   for $k = 2, \ldots, 100$. The error indicator for eigenvalue λ_1 for the Arnoldi method is defined as $\|(I - Q_n Q_n^*)x_1\|$ where $Q_n \in \mathbb{C}^{m\times n}$ is the basis matrix.

   (a) Derive a value α such that (under general conditions which can be ignored)

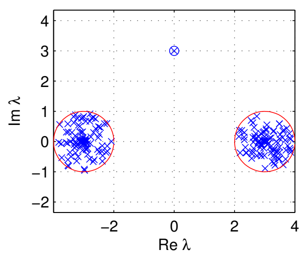
   $$\|(I - QQ^*)x_1\| \le \zeta \alpha^n$$

   where n is the iteration number and x_1 the eigenvector associated with λ_1.

   (b) Give a sharper bound for the error when $n = 101$.

8. * Suppose A is a 201×201-matrix with orthonormal eigenvectors with eigenvalues as in figure.

   Let $C(c, \rho)$ denote a disk centered at c with radius r. Let $(x_1, \lambda_1) = (x_1, 3i)$ be the eigenpair on the imaginary axis. The other eigenvalues satisfy $\lambda_2, \ldots, \lambda_m \in (C(3, 1) \cup C(-3, 1))$. Consider $n = 2n_0$ steps of the Arnoldi method where n_0 integer.



(a) Derive the unique values of ρ and $c \in i\mathbb{R}$ that minimize $\rho/|c-\lambda_1|$ and satisfy $\lambda_2, \ldots, \lambda_m \in C(c, \rho)$.

(b) Derive values for ρ_1, ρ_2, c_1, c_2 such that

$$\left|\frac{c_1 - z}{c_1 - \lambda_1}\right| \left|\frac{c_2 - z}{c_2 - \lambda_1}\right| \leq \frac{4}{9}$$

for all $z = \lambda_2, \lambda_3, \ldots, \lambda_m$. Verify that if

$$p(z) := \left(\frac{c_1 - z}{c_1 - \lambda_1}\right)^{n_0} \left(\frac{c_2 - z}{c_2 - \lambda_1}\right)^{n_0}$$

then $p(\lambda_1) = 1$ and derive a constant α_b such that

$$\max_{\lambda \in \{\lambda_k\}_{k=2}^{n}} |p(\lambda_k)| \leq \alpha_b^{2n_0}.$$

(c) Let the matrix Q be the orthogonal matrix generated by $n = 2n_0$ steps of Arnoldi's method. Both (a) and (b) lead to bounds

$$\|(I - QQ^*)x_1\| \leq \zeta \alpha^n$$

for some value ζ which is independent of n and $\alpha < 1$. Derive values for α for both (a) and (b). Which is better? In this exercise you may use/refer directly to theorems in the course.

(d) Estimate n such that the eigenvector error (in terms of angle) is of order of magnitude of machine precision 10^{-16}. What is the error at $n = 201$? You may neglect rounding errors and assume there is no premature breakdown.

9. These equations define what is called a "discrete-time dynamical system"

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k. \end{aligned}$$

The property called *internal stability* of this equation can be determined from A. If the eigenvalaues satisfy $|\lambda_i| < 1$ for all $i = 1, \ldots, m$ the system is internally stable. If one eigenvalue is larger than one in modulus, $|\lambda_i| > 1$, then the system is unstable. (If there are eigenvalues with $|\lambda_i| = 1$ the analysis is more complicated.)

(a) Suppose $|\lambda_i| \neq 1$, $i = 1, \ldots, m$. Give two conditions which guarantee that the power method for $n \to \infty$ can determine the internal stability of this system.

(b) Suppose the eigenvalues are 0, $2+i3$ and $\frac{1}{2}(1-i)$. What is the convergence factor for the power method for this problem?

10. * The eigenvalue problem can also be reformulated as a nonlinear system of equations with unknowns v and λ for essentially any vector $c \in \mathbb{R}^m \backslash \{0\}$:

$$F(v,\lambda) = \begin{pmatrix} (A-\lambda I)v \\ c^T v - 1 \end{pmatrix} \tag{2}$$

(a) Show that the nonlinear equation (2) is equivalent the eigenvalue problem $Av_* = \lambda_* v_*$ under the condition $c^T v_* \neq 0$.

(b) Show that the Jacobian of F has the structure

$$F'(v,\lambda) = \begin{pmatrix} A-\lambda I & v \\ ?? & ?? \end{pmatrix}.$$

Determine the values for ??.

(c) Show that Newton's method applied to (2) is equivalent to this iteration:

$$u_k = (A-\lambda_k)^{-1} v_k \tag{3a}$$

$$\lambda_{k+1} = ??? \tag{3b}$$

$$v_{k+1} = \frac{1}{\lambda_{k+1}-\lambda_k} u_k \tag{3c}$$

under the assumption that $c^T v_k \neq 0$ for all $k > 1$.

(d) Relate the algorithm in (c) to Rayleigh quotient iteration. What is the convergence order for Newton's method (except for special cases)? What is the generic convergence order for Rayleigh quotient iteration (except for special cases)? Separate between symmetric and non-symmetric A.

11. Suppose $Q_n \in \mathbb{C}^{m\times n}$ and $\underline{H}_n \in \mathbb{C}^{(n+1)\times n}$ are the output of the Arnoldi method applied to the matrix $A \in \mathbb{C}^{m\times m}$. Suppose A is symmetric.

(a) Show that H_n is symmetric and tridiagonal.

(b) Let Q_{n+1} and $\underline{H}_{n+1}$ be the Arnoldi factorization corresponding to the next iterate. Derive a formula for q_{n+1} only involving $w = Aq_n$, q_n, q_{n-1}.

(c) What is the relationship between the Arnoldi method and Lanczos in exact arithmetic?

Exercises about large linear system of equations

12. Explain the relationship between the Arnoldi method and GMRES.

13. Suppose $Q_n \in \mathbb{C}^{m\times n}$ and $\underline{H}_n \in \mathbb{C}^{(n+1)\times n}$ is the output of the Arnoldi method.

    (a) Show that

    $$\min_{x\in\mathcal{K}_n(A,b)} \|Ax-b\|_2 = \min_{z\in\mathbb{C}} \|\underline{H}_n z - \|b\|e_1\|.$$

    (b*) Let $h_{i,j}$ denote the elements of $\underline{H}_n$ and let $H_n \in \mathbb{C}^{n\times n}$ be the upper part of $\underline{H}_n$. Suppose $h_{i+1,i} \neq 0$, $i = 1,\ldots,n-1$ and suppose $h_{n+1,n} = \varepsilon$ and H_n non-singular. Show that $\|Ax_n - b\| \leq \|V\|\|V^{-1}\|\|b\||e_n^T H_n^{-1} e_1||\varepsilon|$.

14. Consider the linear system of equations

    $$\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

    (a) CG is intended for a particular class of problems. Show that this problem satisfies the (sufficient) properties required for the convergence of CG.

    (b) Perform two steps of CG by hand without computer.

    (c) How large is the error in result of (b)?

15. Suppose A is symmetric positive definite. What is the relationship between the approximations generated by GMRES and the approximations generated by CG?

16. State the minimization property of CG (sometimes also used as a definition of CG).

17. Suppose A is invertible and let $x_* = A^{-1}b$. Show that

    $$\|Ax-b\|_{A^{-1}} = \|x - x_*\|_A$$

    for any vector $x \in \mathbb{R}^m$.

18. Let x_i, p_i and r_i denote the vectors generated by the Conjugate Gradient method. Derive a bound

    $$\frac{\|x_n - x_*\|_A}{\|x_0 - x_*\|_A} \leq \min_{p\in P_n^0} \max_i |p(\lambda_i)|.$$

Exercises about dense eigenvalue problems

19. State the basic QR-method.

20. Describe the two-phase approach.

21. Suppose P is a Householder reflector associated with u. Prove:

    (a) $Pu = -u$

    (b) P is Hermitian

22. Given $x \in \mathbb{R}^m$, derive (a formula for) a Householder reflector such that $Pu = \alpha e_k$ for a given k.

23. Let

$$A = \begin{pmatrix} \alpha & \times & \times \\ \beta & \times & \times \\ \gamma & \times & \times \end{pmatrix}$$

    Derive a formula (involving α, β and γ) for an orthogonal matrix Q such that QAQ^T has the structure

$$QAQ^T = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}$$

24. State the shifted QR-method.

25. Suppose A is symmetric. Let $\bar{H}$ be the result of the shifted QR-method with a real shift. Show that $\bar{H}$ is symmetric.

26. * Suppose $A = \begin{pmatrix} a & b \\ \varepsilon & c \end{pmatrix}$. Let $\bar{H}$ be the result of the shifted QR-method with shift c. Show that the shifted QR-method is quadratic with respect to ε. More precisely, show that $\bar{H} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \mathcal{O}(\varepsilon^2) & \tilde{c} \end{pmatrix}$.

27. * Suppose $A = \begin{pmatrix} a & \varepsilon \\ \varepsilon & c \end{pmatrix}$. Let $\bar{H}$ be the result of the shifted QR-method with shift c. Show that the shifted QR-method is cubic with respect to ε. More precisely, show that $\bar{H} = \begin{pmatrix} \tilde{a} & \mathcal{O}(\varepsilon^3) \\ \mathcal{O}(\varepsilon^3) & \tilde{c} \end{pmatrix}$. (Difficult)

28. In what sense is the convergence demonstrated in exercise 26 and 27 consistent to the convergence of the convergence of the Rayleigh quotient iteration?

Exercises about functions of matrices

29. State the Taylor definition of matrix functions.

30. State the Jordan form definition of matrix functions.

31. Let J be a Jordan block corresponding to eigenvalue λ. Show that the Taylor definition and Jordan definition are equivalent for this matrix.

32. Use the following MATLAB output and compute an approximation of $\exp(A)$, as the result of scaling-and-squaring where the initial approximation is computed with a truncated Taylor series.

```
>> B=eye(n)+A/(16)+A^2/32;
>> BB=B^2
BB=
   1.18380  -0.12917   0.10335
  -0.19689   1.27569   0.26426
   0.16304  -0.15383   1.02728
>> BB=BB^2
BB=1.44367  -0.33358   0.19438
  -0.44115   1.61216   0.58824
   0.39077  -0.37531   1.03150
>> BB=BB^2
   2.30731  -1.09233   0.28490
  -1.11823   2.52545   1.46934
   1.13279  -1.12255   0.91917
>> BB=BB^2
BB =
   6.86790  -5.59880  -0.68577
  -3.73966   5.94995   4.74273
   4.91020  -5.10415  -0.48180
>> BB=BB^2
BB =
   64.738  -68.264  -30.933
  -24.647   32.132   28.499
   50.445  -55.402  -27.343
```

33. Use the interpolation property of matrix functions to derive an explicit formula for

$$\exp\begin{pmatrix}1 & \alpha\\ 0 & 2\end{pmatrix}.$$

34. Suppose $T_1, T_2 \in \mathbb{R}^{n\times n}$ are two triangular matrices and the diagonal elements of T_1 are equal to the diagonal elements of T_2. Show that there exist matrices $A_0, \ldots, A_{n-1}$ such that

$$f(\alpha T_1 + (1-\alpha)T_2) = \sum_{i=0}^{n-1} \alpha^i A_i$$

where A_i depends on T_1, T_2 and f (but not α).

35. Suppose $B^T = -B$.
    (a) Show that if λ is an eigenvalue of B, then $-\lambda$ is an eigenvalue of B.
    (b*) Show that $Q = \exp(B)$ is an orthogonal matrix.

36. Describe the method called Krylov method for matrix functions.

37. Let T be a triangular matrix with distinct eigenvalues and let

$$f(T) = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{13} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}.$$

    (a) Give the values for f_{11}, f_{21}, f_{31}, f_{22}, f_{32}, f_{33}.
    (b) Derive an explicit formula for f_{12} involving only elements of T and the values computed in (a).

2 Hints/solutions for selected exercises

4 c. Let $\Delta v = v - v_*$.

$$\begin{aligned} v^*v = (v_* + \Delta v)^*(v_* + \Delta v) = v_*^*v_* + \Delta v^*v_* + v_*^*\Delta v + \Delta v^*\Delta v = \\ v_*^*v_* + \Delta v^*v_* + \overline{\Delta v^*v_*} + \Delta v^*\Delta v = \\ v_*^*v_* + 2\,\mathrm{Re}(\Delta v^*v_*) + \Delta v^*\Delta v = \end{aligned}$$

$$v^*Av = (v_*+\Delta v)^*A(v_*+\Delta v) = v_*^*Av_*+\Delta v^*Av_*+v_*^*A\Delta v+\Delta v^*A\Delta v$$

8 b-c. Parameterize with respect to $c = -i\beta$. Smallest disk has radius $\rho = 1 + \sqrt{3^2+\beta^2}$. Hence $\alpha = \rho/|c-\lambda_1| = (1+\sqrt{\beta^2+3^2})/(\beta+3)$. $\alpha'(\beta) = 0 = \sqrt{\beta^2+9} - 3\beta + 9 \Rightarrow \beta_* = \frac{3\cdot\sqrt{17}+27}{8} \Rightarrow \alpha_* \approx 0.8539.$)

10c Iterates of Newton's method satisfy

$$F'(v_k, \lambda_k)\begin{pmatrix} v_{k+1} - v_k \\ \lambda_{k+1} - \lambda_k \end{pmatrix} = -F(v_k, \lambda_k).$$

First consider the last row and prove that $c^Tv_{k+1} = 1$ for any k. Subsequently multiply the first block row with $c^T(A-\lambda_kI)^{-1}$ and use that $c^Tv_k = 1$.

13b Hint: Use the Arnoldi factorization and that $\|Ax_n - b\| = \|AQ_n z_n - b\| = \|Q_n \underline{H}_n z_n - b\|$.

26-27 Derive an explicit formula for the QR-decomposition of an arbitrary 2×2-matrix. This can be done with Givens rotations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = QR$$

where

$$Q = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

and $\alpha = a/\sqrt{a^2+c^2}$, $\beta = c/\sqrt{a^2+c^2}$. The R-matrix can be computed with $R = Q^T A$. In order to compute the (2,1)-element of the result of one step of the shifted QR-method you need to compute the last row of R and the first column of Q for the shifted A-matrix.

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