0 Fundamental concepts

This is a short summary of certain important fundamental techniques, which are not a part of the course but will be used several times.

0.1 Orthogonal matrices and QR factorization

The QR-factorization is a factorization involving an orthogonal matrix Q and an upper triangular matrix R.

Orthogonal matrices are important in matrix computations, most importantly when the matrix represents a basis of a vector space. Many properties of the vector space are not robust with respect to rounding errors if the basis is not orthogonal.

Definition 0.1.1 (Orthogonal matrix)**.** *$Q \in \mathbb{R}^{n\times m}$ is called an orthogonal matrix if*

$$Q^T Q = I.$$

For complex matrices, the corresponding property is called unitary: $Q^* Q = I$.

Properties:

(i) The columns of an orthogonal matrix are orthonormal.

(ii) If $n = m$, then $Q^T = Q^{-1}$.

(iii) If $n = m$, then $QQ^T = I$.

Note that (iii) is not satisfied if Q is a rectangular matrix ($n \neq m$). For instance

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is an orthogonal matrix since $Q^T Q = I$, but

$$QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 0.1.2 (Uniqueness of QR-factorization)**.** *For any matrix* $A \in \mathbb{R}^{n\times m}$, $n \leq m$ *there exists an orthogonal matrix* $Q \in \mathbb{R} \in \mathbb{R}^{n\times m}$ *and an upper triangular matrix* $R \in \mathbb{R}^{m\times m}$ *such that*

$$A = QR.$$

Moreover, if A *is non-singular, the diagonal elements of* R *can be chosen positive, and the decomposition where the diagonal elements of* R *are positive is unique.*

A QR-factorization can be computed with the matlab command `[Q,R]=qr(A)`. It will however in general not return the solution with positive diagonal elements.

The QR-decomposition is the underlying method to solve overdetermined linear systems of equations, for instance with the backslash operator in matlab when the matrices are rectangular. The decomposition can be computed in a finite number of operations with for instance Householder reflectors or Givens rotations, which will be used in this course.

0.2 *Jordan canonical form (JCF)*

The Jordan canonical form, also sometimes the Jordan form or the Jordan decomposition, is a transformation that brings the matrix to a certain block diagonal form.

The Jordan canonical form is typically treated in basic linear algebra courses, but also used in the study of stability theory in for instance systems theory and differential equations.

Definition 0.2.1 (Jordan canonical form)**.** *The Jordan decomposition of a matrix* $A \in \mathbb{R}^{n\times n}$ *is an invertible matrix* $X \in \mathbb{R}^{n\times n}$ *and a block diagonal matrix*

$$D = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_i\times n_i}$$

such that

$$A = XDX^{-1}.$$

The matrix J_i *is called a Jordan block, or sometimes a Jordan matrix.*

Example of Jordan decomposition

The Jordan decomposition of

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

is represented by

$$D = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

since $A = XDX^{-1}$. Note that the eigenvalue $\lambda = 3$ has two Jordan blocks, one of size $n_1 = 2$ and one of size $n_2 = 1$.

Theorem 0.2.2 (Existance and uniqueness of Jordan decomposition)**.** *All matrices $A \in \mathbb{R}^{n\times n}$ have a Jordan decomposition. The (unordered) set of Jordan blocks is unique for any given matrix.*

Definition 0.2.3. *If a matrix A has a Jordan decomposition where all Jordan blocks have size one (such that D is a diagonal matrix) the matrix A is called a diagonalizable matrix.*

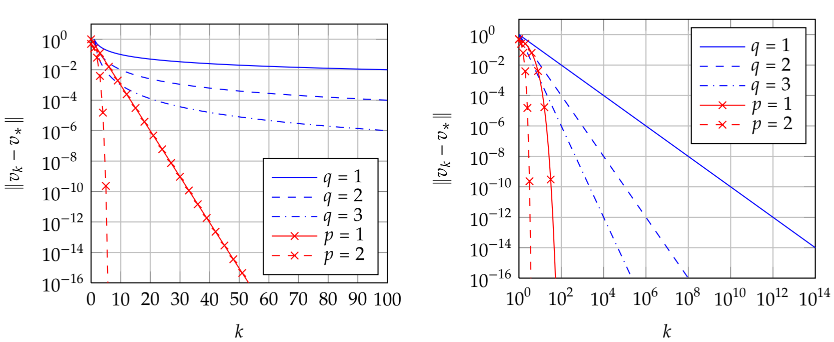
Lemma 0.2.4. *Suppose A is symmetric. Then A is diagonalizable and the X matrix in the Jordan decomposition can be chosen as an orthogonal matrix.*

Lemma 0.2.5. *If $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues, then A is diagonalizable.*

0.3 Types of convergence

Suppose $v_k \in \mathbb{R}^n$, $k = 1, \ldots$ is a sequence of vectors such that that v_k converges to some vector $v_* \in \mathbb{R}^n$ as $k \to \infty$. In our context, k corresponds to an approximation parameter in a numerical method. For instance, it can correspond to the number of grid points in a trapezoidal rule or the iteration number in Newton's method.

We separate between two types of convergence. One type of convergence is characterized with q, called *the order of accuracy*, and the other with p which is called *the convergence order*. The same error curves are given in semilogarithmic and logarithmic scale below.



The figures shows $e_k = k^{-q}$ and $e_{k+1} = \frac{1}{2}e_k^p$ respectively. Note that the figures show the same curves with different scaling of the x-axis.

A note of caution: The term "linear convergence" can mean different things depending on context. The meaning is mostly field dependent. In the context of differential equations linear convergence typically refers to order of accuracy with $q = 1$. In the context of iterative methods, linear convergence typically refers to convergence order with $p = 1$.

In this course we are mainly concerned with iterative methods and will mostly study the convergence order. The convergence order is better visualized in semilogy-plots (left).

0.3.1 *Algebraic convergence and order of accuracy*

The order of accuracy corresponds to the situation when the error behaves as $\sim k^{-q}$.

Definition 0.3.1 is a classification of a sequence of approximations. We will analogously say that a numerical method has order of accuracy q if it generically generates a sequence of approximations which has order of accuracy q.

Definition 0.3.1 (Order of accuracy)**.** *The sequence of approximations v_k, $k = 0, \ldots$, has order of accuracy q, if*

$$\|v_k - v_*\| \leq Ck^{-q}, \tag{1}$$

for sufficiently large k for some constant C, where q is the largest value such that (1) is satisfied.

Some common examples:

- The trapezoidal rule for the numerical computation of integrals has order of accuracy $q = 2$, where we normally define the step-length as $h = (b - a)/k$ such that $k^{-2} = \mathcal{O}(h^2)$.
- The forward and backward Euler method for to solve an initial value problem to a given time-point has order of accuracy $q = 1$.
- The truncated Fourier expansion of a discontinuous function has order of accuracy $q = 1$ in an ℓ_2-sense.

The convergence in Definition 0.3.1 is sometimes called *algebraic convergence*.

0.3.2 *Exponential convergence and convergence order*

The term *convergence order* relates the error between two consecutive approximations.

Definition 0.3.2 (Convergence order)**.** *The sequence of approximations v_k, $k = 0, \ldots$, has convergence order p if*

$$\frac{\|v_{k+1} - v_*\|}{\|v_k - v_*\|^p} \leq C < \infty \tag{2}$$

for sufficiently large k for some constant C. The smallest value C such that (2) is satisfied is called the convergence factor (corresponding to convergence order p).

There is a more explicit characterization for the special case $p = 1$. For $p = 1$ we have

$$\|v_{k+1} - v_*\| \leq C\|v_k - v_*\|,$$

which by induction can be shown to be equivalent to

$$\|v_k - v_*\| \leq C^k\|v_0 - v_*\|. \quad (3)$$

The type of convergence in (2) is always asymptotically faster than (1) and (2) is said to have *infinite order of accuracy*.

Some common examples:

- Newton's method has in general convergence order $p = 2$, if the root is simple. For repeated roots the convergence order is $p = 1$.
- The secant method has, in general, convergence order $p = (1 + \sqrt{5})/2 \approx 1.62$.
- The Gauss-Seidel iteration for linear systems of equation has convergence order $p = 1$.
- Fixed point iterations $v_{k+1} = \varphi(v_k) \in \mathbb{R}^n$ in general has convergence order $p = 1$ if $\varphi(v_*) \neq 0$. The convergence factor is given by $\max(|\lambda(\varphi'(v_*))|)$, where φ' is the Jacobian of φ.

Similar to the way algebraic convergence refers to the order of accuracy, convergence of the type (2) is often called *exponential convergence* or sometimes *geometric convergence*. The terminology is however not unified. In some literature geometric convergence refers to $p = 1$ and $p > 1$ would refer to supergeometric convergence. For any meaning of the word, geometric convergence is asymptotically faster than algebraic convergence.