



## Homework 1

Information about homework: If correct solutions to all questions are handed in before the deadline, two bonus points will be awarded to the final written exam. If a solution that is handed in before that date is not correct, it has to be redone, but the second time without yielding bonus points for the exam. Solutions must be clearly written, and easy to follow. If not, they will not generate bonus points, and must be redone. It is highly recommended that the answers are done on a computer (for instance LaTeX or word).

Please submit, not later than midnight of the date of the deadline:

- a written report with the solutions (hard-copy) during the lecture or in the “homework mailbox” next to the mathematics studentexpedition on Lindstedsvägen 25; and
- the source code of your programs by e-mail to the lecturer (eliasj at kth.se).

Deadline Homework 1: Nov. 12, 2015

1. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 9 \end{pmatrix}.$$

Plot the eigenvalue error as a function of iterate for the following algorithms in a semilog plot, and relate with the convergence theory.

- Power method with the starting vector  $x_0^T = (1, 1, 1)$ .
  - Rayleigh quotient iteration with the starting vector  $x_0^T = (1, 1, 1)$ .
  - Change the entry of the matrix in element row 1, column 3, to  $a_{1,3} = 4$  and again run the Rayleigh quotient iteration with  $x_0^T = (1, 1, 1)$ . Explain why we obtain slower convergence in comparison to (b).
2. In this exercise we will investigate the performance of different versions of the Gram-Schmidt orthogonalization when it is combined with the Arnoldi method. Consider the matrix  $A$  constructed with the following command

```
rand('seed',0); A=gallery('wathen',nn,nn); (1)
```

You may in this exercise use `eig(A)` as a reference/exact solution.



(a) Modify the orthogonalization in `arnoldi.m` available from the course web page. Apply it to (1) and generate the values in the following table, where `time` is the CPU-time and `orth` =  $\|Q_m^T Q_m - I\|$  is an indicator of the orthogonality of the basis, and  $m$  the number of iterations in Arnoldi's method.

Use MATLAB-commands `tic` and `toc` to measure CPU-time.

	single GS		modified GS		double GS		triple GS	
	time	orth	time	orth	time	orth	time	orth
$m = 5$								
$m = 10$								
$m = 20$								
$m = 50$								
$m = 100$								

Select  $n \geq 10$  such that the maximum computation time for  $m = 100$  is approximately 10 minutes on your computer.

(b) Interpret the result in (a). Is one version "best" in this setting? If so, in what sense is it "best"? Discuss the fairness of the comparison.

Background Galerkin method: The Galerkin method is a fundamental technique in the finite element method for partial differential equations. The approximation is generated from the bilinear form  $a(\cdot, \cdot)$  and  $f(\cdot)$  on vector space  $V = \text{span}(v_1, \dots, v_m)$  and is defined by the equality

$$\sum_{i=1}^m w_i a(v_i, v_j) = f(v_j)$$

for all  $v_j, j = 1, \dots, m$ .

3. We shall here investigate a primitive variant of the Arnoldi method. Let  $K_m$  be a matrix with the iterates of the power method, that is, let  $K_m := [b, Ab/\|Ab\|, \dots, A^{m-1}b/\|A^{m-1}b\|] \in \mathbb{R}^{n \times m}$ .

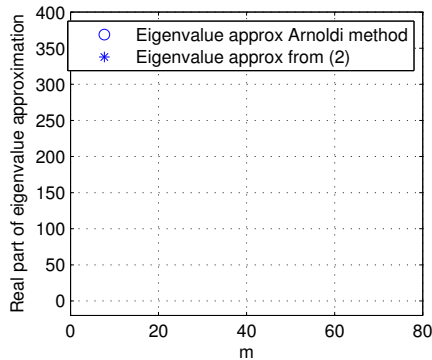
(a) The approximation stemming from Galerkin method applied to the bilinear form associated with the eigenvalue problem ( $a(u, v) = u^T Av - \mu u^T v, f(v) = 0$ ) for the subspace spanned by the columns of  $K_m$  satisfies

$$\mu K_m^T K_m w = K_m^T A K_m w \tag{2}$$

Hint: Recall that if  $V_m$  is an orthogonal basis of  $\mathcal{K}_m(A, b)$ , then the Arnoldi approximation is eigenvalues of  $V_m^T A V_m$ . Maybe there exists a non-singular matrix  $R$  such that  $K_m = V_m R$ ? Show that  $R^T R = K_m^T K_m$  by considering  $V_m^T V_m$ .

You may assume that  $\mathcal{K}_m(A, b)$  has dimension  $m$ . Prove that the approximation (2) is identical to the approximation generated by Arnoldi's method for eigenvalue problem. No prior knowledge about the Galerkin method is required to solve the problem.

(b) By solving (2) directly with `eig`, we have a method which requires only computation of  $K_m^T K_m$  and  $K_m^T A K_m$ , and does not require any orthogonalization. Apply this method to (1) and compare with the Arnoldi method. Use double GS as orthogonalization. Generate the following figure for  $m = 1, 2, \dots$ : Use a random starting vector, with the same starting vector for all simulations.



(c) Interpret the result of (a)-(b). What do we expect in exact arithmetic? What do we observe? Which approach is better? Why?

4. Download and load the matrix

<http://www.math.kth.se/na/SF2524/matber14/Bwedge.mat>

Use the command `load('Bwedge.mat')` to access the matrix  $B$  and eigenvalues.

(a) Accurate estimations of the eigenvalues are given in `B_eigvals`. Plot these and use the convergence theory for Arnoldi method for eigenvalue problems to

- identify which eigenvalues will be approximated first, and
- estimate a convergence factor of the eigenvector angle (by estimating circles in a plot with the eigenvalues) corresponding to the first three eigenvalues that will be approximated.

The convergence factor is constant  $\alpha$  such that  $\text{error} \sim \alpha^n$ .

(b) Generate figures with the Ritz values for the Arnoldi method with a random starting vector, and verify and relate with (a) for  $m = 2, 4, 8, 10, 20, 30, 40$ .

(c) Repeat (b) with shift-and-invert Arnoldi method instead of Arnoldi method. Use shift  $\sigma = -11 + 2i$ . Carry out an LU-factorization of  $C = B - \sigma I = LU$  and modify the reference implementation `arnoldi.m` such that in every step you need to solve one upper-triangular and one lower-triangular system of equations. Plot the approximations of  $B$  and relate to the eigenvalues of  $C^{-1}$ .

Use MATLAB command `[L,U,P]=lu(C)` to compute a (variant of a) LU-factorization. Note that this function computes a factorization  $PC = LU$  such that  $C \setminus z = U \setminus (L \setminus (P * z))$ .

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Only for PhD students taking the course *SF3580 Numerical linear algebra*:

5. Exercise about restarting. Use the matrix `Bwedge.mat` in the simulations.

(a) Implement an explicit restarting strategy for the Arnoldi method as follows. Take a linear combination (with unit weights) of the Ritz vectors generated by  $m$  iterations of the Arnoldi method as a new starting vector. Use the Ritz vectors corresponding to the  $k$  largest Ritz values. Extract the Ritz vectors as follows:



```
[Q,H]=arnoldi(A,b,m);  
[V,D]=eig(H(1:end-1,1:end)); d=diag(D);  
[Y,I]=sort(-abs(d));  
ritz_vals=d(I(1:k));  
ritz_vecs=Q(:,1:end-1)*V(:,I(1:k));
```

Carry out the experiments for  $m = 10, k = 5$  and  $m = 20, k = 10$  with 100 restarts. Plot the eigenvalue approximation as a function of restart. Does the restart strategy work in practice?

- (b) Use the reference code `arnupd.m` and `arnoldi_sorensen.m` which are implementations of an implicit restarting strategy. Run the algorithm with same restarting parameters as in (a) and explain the difference. Implicit restarting is in general to prefer over explicit restarting. Why?

The programs `arnoldi_sorensen.m` and `arnupd.m` are reference implementations of *Implicit application of polynomial filters in a k-step arnoldi method*, D. C. Sorensen, *SIAM J. Matrix Anal. Appl.* Vol. 13, No. 1, pp. 357-385, 1992. It is not necessary to read the details of the paper to solve the homework. The programs are available from <http://www.math.kth.se/~eliasj/NLA/arnupd.m> Another good reference for implicit restarting are provided in the lecture notes: <http://people.inf.ethz.ch/arbenz/ewp/Lnotes/chapter10.pdf>