

SF2524 Matrix Computations for Large-scale Systems
Exam

Aids: None **Time: Four hours**

Grades: E: 14 points, D: 16 points, C: 18 points, B: 20 points, A: 22 points (out of the possible 27 points, including bonus points from homeworks).

Problem 1 (3p)

The program to the right corresponds to GMRES and CG as we have implemented them in the course: Q, H correspond to an Arnoldi factorization. X, P, R are matrices containing the vectors computed throughout CG.

```
>> size(A)
ans =
   1000    1000
>> N=3;
>> [Q,H,x_gmres]=mygmres(A,b,N);
>> [x_cg,rhist,X,P,R]=cg(A,b,N);
>> Z=inv(Q(:,1:N)'*X);
>> f=norm(Q(:,1:N)-X*Z)
```

(a) What will f be in the output? Explain your reasoning and any simplifying assumption.

(b) Suppose the eigenvalues are contained in a disk of radius 5 centered at 10. Give a bound on $\|Ax - b\|/\|b\|$ where `x=x_gmres`.

Problem 2 (4p)

(a) Describe (or derive) the method called Newtons method for the matrix square root.

(b) Generalize the method and propose a method for the matrix function $F = f(A)$

$$A = \exp(A) + F$$

where A is a symmetric matrix.

Problem 3 (5p)

(a) Describe the Arnoldi relation? What are the matrices and the non-zero structure of the matrices?

(b) Suppose the Krylov subpace with starting vector c contains an eigenvector of the matrix. Prove that this eigenpair will be computed exactly with Arnoldi's method for eigenvalue problems if it is started with c.

(c) Prove that GMRES converges in at most $k = n$ steps.

Problem 4 (4p) Suppose the function f is piecewise constant and

$$f(x) = \begin{cases} -1 & x < -1 \\ 0 & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

and $A = QDQ^T$ where $D = \frac{1}{2}\operatorname{diag}(-5, -2, \ldots, 1, 2, 5)$ where Q is an orthogonal matrix.

(a) Which matrix functions definitions are applicable extensions to matrix functions for this matrix A? Jordan-definition, Taylor definition and/or Cauchy integral definition?

(b) What is $f(A)$ if D is replaced by $D = \frac{1}{6}\operatorname{diag}(-5, -2, \ldots, 1, 2, 5)$?

Problem 5 (4p) Does the basic QR-method converge, and if so in approximately how many iterations does it converge? Use termination criteria $\|Q^{(k)} - X\| \leq 10^{-10}$ where $XRX^T = A$ is a Schur factorization. Specify any simplifying assumptions.

(a)

$$A = \begin{bmatrix} 1 & 2 & A_{1,3} \\ & 3 & A_{2,3} \\ & & A_{3,3} \end{bmatrix} \quad \text{where} \quad A_{3,3} = \begin{bmatrix} \pi & \alpha \\ & \pi + \varepsilon \end{bmatrix}$$

(b) $A = Q \in \mathbb{R}^{10 \times 10}$ where Q is an orthogonal matrix.

(c) A symmetrix matrix with eigenvalues $1, 3, 5$.

Problem 6 (4p) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix.

(a) Describe the Rayleigh quotient iteration for this matrix.

(b) What is the relationship between Rayleigh quotient iteration and the power method?

(c) Prove that if the starting vector x_0 in Rayleigh quotient iteration is orthogonal to the eigenvector v_1 then all iterates will be orthgonal to v_1 and Rayleigh quotient iteration will therefore not converge to the corresponding eigenpair (λ_1, v_1).