SF2524 Matrix Computations for Large-scale Systems
Exam

Aids: None Time: Four hours

Grades: E: 14 points, D: 16 points, C: 18 points, B: 20 points, A: 22 points (out of the possible 27 points, including bonus points from homeworks).

Problem 1 (4p)

(a) What is an Arnoldi factorization? Describe the properties of the matrices involved in the factorization.

(b) Suppose r_A is the Rayleigh quotient for A and suppose r_H is the Rayleigh quotient for H_m. Find a function f such that $r_H(z) = f(r_A(Q_m z))$ for all z. In other words, express $r_H(z)$ in terms of $r_A(Q_m z)$

(c) Suppose the eigenvector approximation $\tilde{x} = Q_m z$ has accuracy $\mathcal{O}(\alpha^m)$ for some small value α. What is the order of magnitude of the accuracy of the eigenvalue approximation $r_A(Q_m z)$, if A is symmetric?

Problem 2 (4p)

(a) Describe the basic QR-method, in formulas or simple MATLAB-code.

(b) Describe the shifted QR-method, in formulas or simple MATLAB-code.

(c) Suppose $A \in \mathbb{R}^{n\times n}$ satisfies the property $A = \rho A^T$ for some value ρ. Show that this property is preserved by the QR-method. Under what conditions is it preserved for the shifted QR-method?

Problem 3 (3p) Prove the following generalization of the min-max theory for the convergence of GMRES. Suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix and suppose $b = c_1 v_1 + \cdots + c_\ell v_\ell$ with $\ell < n$, and $v_1, \ldots, v_\ell$ are normalized eigenvectors of A. The constants $c_1, \ldots, c_\ell$ are non-zero and satisfy $c_1^2 + \cdots + c_\ell^2 = 1$. Show that

$$\|Ax_k - b\| \leq \min_{p\in P_k^0} \max_{i=1,\ldots,\ell} |p(\lambda_i)|.$$

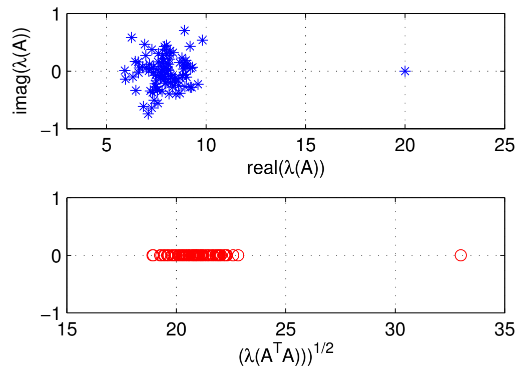
where x_k is the iterate generated by k steps of GMRES. You may use any theorem in the course.
Hint: You may use that symmetric matrices are diagonalizable and that the eigenvectors are orthogonal.

Problem 4 (5p)

Suppose the eigenvalues and singular values of a matrix A are as in the figure to the right. Give a constant $\beta < 1$ such that after k iterations we have

$$\text{err} \leq \alpha\beta^k.$$

for method ♣. Define also how the error *err* is measured. You may invoke any theorem in the course. Answer this for ...



(a) ♣ = GMRES

(b) ♣ = CGN (sometimes called CGNE)

(c) ♣ = Arnoldi's method for eigenvalues corresponding to the eigenvalue close to $\lambda_0 \approx 20$.

Problem 5 (4p) Suppose $A \in \mathbb{R}^{n\times n}$ is a lower triangular matrix. (Note: Not upper triangular) Let f be an analytic function and let $F = f(A)$ be the corresponding matrix function.

(a) What is in general the non-zero structure of F?

(b) Provide a derivation of a formula for f_{ij} only involving the ith and jth row and column of A and F.

(c) How can the formula be used to construct an algorithm for the matrix function of a lower triangular matrix?

Problem 6 (4p)

(a) What is a φ-function and how can the matrix function $\varphi(A)$ be used to solve ordinary differential equations?

(b) How is the Krylov approximation f_m of $\varphi(A)b$ constructed from the Arnoldi factorization? If a small matrix function has to be computed, propose a procedure.

(c) In this course a theorem stated that (under appropriate assumptions on the matrix A) the Krylov approximation f_m satisfies

$$\|f_m - f(A)b\| \leq \alpha \min_{p\in P_{m-1}} \max_{i=1,\ldots,n} |f(\lambda_i) - p(\lambda_i)|.$$

Suppose A has eigenvalues λ_i, $i = 1,\ldots,n$ such that $|\lambda_i| < 1/2$. Specialize the formula and bound it with an explicit formula involving m showing that the approximation error goes to zero very fast. Problem (c) can be answered without answering (a)-(b).
Hint: You may find it useful that the remainder term of the Taylor expansion at zero of the φ-function satisfies: $|R_N(x)| \leq c|x|^N/N!$.