ON THE NONLINEAR DYNAMICS OF FAST FILTERING ALGORITHMS*

CHRISTOPHER I. BYRNES†, ANDERS LINDQUIST‡, AND YISHAO ZHOU‡

ABSTRACT. The main purpose of this paper is to address a fundamental open problem in linear filtering and estimation, viz what is the steady-state or asymptotic behavior of the Kalman filter, or the Kalman gain, when the observed stationary stochastic process is not generated by a finite-dimensional stochastic system, or when it is generated by a stochastic system having higher dimensional unmodelled dynamics. For example, some time ago Kalman pointed out that the usual positivity conditions assumed in the classical situation are not in fact necessary for the Kalman filter to converge. Using a "fast filtering" algorithm, which incorporates the statistics of the observation process as initial conditions for a dynamical system, this question is analyzed in terms of the phase portrait of a "universal" nonlinear dynamical system. This point of view has additional advantages as well, since it enables one to use the theory of dynamical systems to study the sensitivity of the Kalman filter to (small) changes in initial conditions; e.g. to changes in the statistics of the underlying process. This is especially important since these statistics are often either approximated or estimated. In this paper, for a scalar observation process we derive necessary and sufficient condition for the Kalman filter to converge, using methods from stochastic systems and from nonlinear dynamics - especially the use of stable, unstable and center manifolds. We also show that, in nonconvergent cases, there exist periodic points of every period p, $p \geq 3$ which are arbitrarily close to initial conditions having unbounded orbits, rigorously demonstrating that the Kalman filter can also be "sensitive to initial conditions".

Key words. Kalman filtering, fast filtering algorithms, Riccati equations, nonlinear dynamics, dynamical systems, power method, Lagrange-Grassmannian manifolds

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1. Introduction

Given a scalar stationary stochastic process $\{y_0, y_1, y_2, \dots\}$ which is the output of a linear, finite dimensional stochastic system driven by white noise, it is well-known that the minimum variance estimate $\hat{x}_t$ of the current state x_t of the system is generated by the Kalman filter. Indeed, the Kalman filter is a model of the unforced stochastic system driven by a term consisting of the current output estimation error amplified by the so-called "Kalman gain" k_t, which itself can be determined "off-line" by solving a matrix Riccati equation. In this case, the steady-state behavior of both the Riccati equation and the Kalman filter is well understood. The purpose of this paper is to address a fundamental open problem concerning filtering and estimation, viz what is the steady-state or asymptotic behavior of the Kalman filter, or the Kalman gain, when the stochastic process $\{y_t\}$ is not generated by a stochastic system, or when it is generated by a stochastic system having higher dimensional, unmodelled dynamics? This question has been raised, for example, by Kalman who

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† Department of Systems Science and Mathematics, Washington University, St. Louis, Missouri 63130, USA.

‡ Division of Optimization and Systems Theory, Royal Institute of Technology, 100 44 Stockholm, Sweden.

pointed out that the positivity constraints associated with the existence of a stochastic system realizing $\{y_t\}$ might not be necessary for the Kalman filter to converge, a fact rigorously established for first-order systems in [5] and for two dimensional systems in [6]. Indeed, in [5] a complete phase portrait of the Kalman gain and the Kalman filter, as a dynamical system, was derived for first-order systems.

The basis for this analysis of the Kalman filtering as a dynamical system was the formulation [25] of "fast filtering" algorithms two decades ago. Instead of determining the n-vector k_t by first solving a matrix Riccati equation for a symmetric matrix P_t, involving $n(n+1)/2$ variables, the fast filtering algorithm involves solving only a system of $2n$ equations, which consist of a dynamical system propagating k_t and an "adjoint" vector k_t^*. Moreover, as first shown in [27] and crucial for our dynamical systems analysis of the Kalman filter, this dynamical system can be reformulated so that the statistics of the process $\{y_t\}$ enter into the fast filtering algorithm only as initial conditions. Thus, one can analyze the asymptotic behavior of the Kalman filter for different statistics in terms of the phase portrait of the fast filtering algorithm. This is in sharp contrast to analysis of the Riccati equation as a dynamical system, since different statistics lead to different Riccati equations and, in fact, not to different initial conditions.

This point of view has additional advantages as well, since it enables one to study the sensitivity of the Kalman filter to (small) changes in initial conditions; e.g. to changes in the underlying system $\{y_t\}$ or its statistics. This is especially important since the statistics of the underlying process are often either approximated or estimated. In this direction, for the first-order case necessary and sufficient condition for asymptotic convergence of k_t were discovered [5], verifying the expectation that the Kalman filter would indeed converge for a much larger set of initial conditions or "initial statistics" than the classical theory predicts. In the complement of this set of (convergent) initial conditions, it was shown that there existed infinitely many periodic points of each period p, $p \geq 3$. Moreover, arbitrarily close to each of these periodic initial conditions are initial conditions for trajectories which are unbounded. For this reason, in the complement of the set of convergent initial conditions the Kalman filter is sensitive to initial conditions.

In this paper, for n:th order filtering problems we derive a systems theoretic necessary and sufficient condition on the process $\{y_t\}$ for the sequence of Kalman gains, k_t, to converge to a classical limit. En route to this result, we need to develop a good understanding of the phase portrait of the fast filtering algorithm as a nonlinear dynamical system, including the determination (via spectral factorization) of a complete set of analytic invariant integrals. This, in turn, requires the extension of the several classical and more recent results concerning positive real transfer functions, positive semidefinite Toeplitz forms, and spectral factorization to situations where the relevant positivity conditions are not necessarily satisfied. Indeed, one of the main themes of this paper is that several important results classically conceived in terms of certain positivity conditions actually hold in a more universal context. While our main interest in this phenomenon lies in characterizing when the Kalman filter converges to a classical limit, this theme is of course quite old. For example, Hurwitz's derivation [20] of the Routh-Hurwitz criterion, actually computed the difference between left-half plane and right-half plane zeros (or poles) as the signature of a Hankel matrix, while the Routh-Hurwitz conditions are simply the inequalities reflecting the positivity of this Hankel matrix. A more recent, and more relevant, example is the relaxation of the positive real conditions in circuit synthesis in the development of modern realization theory, based on rationality of transfer functions, or on rank conditions on Hankel matrices.

The paper is organized as follows. In Section 2, we set notation and recall some preliminary results needed throughout the paper. We begin Section 3 by reviewing some of the important relationship between shaping filters and Toeplitz forms, both for positive real transfer functions and in general. This relationship then enables us to extend an elegant parameterization, discovered by Kimura [22] and by Georgiou [16], of positive real transfer functions in terms of Szegö polynomials to a parameterization of all rational transfer functions. Just as the Kimura-Georgiou parameterization plays an important role in the covariance extension problem, this generalized parameterization plays an essential role in analyzing the global asymptotic behavior of the Kalman filter. This generalized Kimura-Georgiou parameterization is in fact a *bona fide* (birational) change of coordinates, as we show in Section 4. We then express the fast filtering algorithm and (what turns out to be a complete set of) its analytic invariant integrals in this new coordinate system. We begin Section 5 with a brief introduction to stable, unstable and center manifold theory and its application to local stability analysis. After calculating the dimensions of these invariant manifolds at an equilibrium of the fast filtering algorithm, we show that the level sets of the invariant integrals defined above locally define smooth submanifolds near the equilibria and we identify the invariant manifolds in terms of these invariants.

In Sections 6 - 7 we turn to the problem of global convergence of the fast filtering algorithm. In terms of the basic invariant integrals it is easy to determine a system theoretic necessary condition for an initial condition to generate a trajectory of the fast filtering algorithm, which converges to a classical limit. This condition, derived from a spectral factorization argument, is simply that a certain pseudo-polynomial be sign-definite on the unit circle. Moreover, the local stability analysis carried out in Section 5 shows that, for initial conditions sufficiently near an equilibrium, this necessary condition is also sufficient locally. Our main result, Theorem 7.1, asserts that this is also true in the large: except for a thin set of points which escape in finite time (and which can be explicitly characterized) a necessary and sufficient condition for global convergence of the Kalman gain k_t to a limit k_∞ is sign definiteness of the corresponding pseudo-polynomial. The proof is based on a well-known interpretation of fast filtering algorithms and an equivalent Riccati equation as a dynamical system evolving on a Lagrangian Grassmannian. We conclude the paper in Section 8, with a series of examples and simulations for first and second order systems.

2. **Preliminaries**

Let $v(z)$ be a proper rational function of degree n with a minimal realization

$$v(z) = \frac{1}{2} + h'(zI - F)^{-1} g \tag{2.1}$$

(where $F \in \mathbb{R}^{n\times n}, g, h \in \mathbb{R}^n$ and prime denotes transpose) and consider the corresponding matrix Riccati equation

$$P_{t+1} = FP_tF' + (g - FP_th)(1 - h'P_th)^{-1}(g - FP_th)' \tag{2.2}$$

having an orbit of symmetric matrices $\{P_1, P_2, P_3, \dots\}$ for each symmetric $P_0 \in \mathbb{R}^{n\times n}$,

If $v(z)$ is *positive real*, i.e.

$$\text{(i)} \qquad v(z) \text{ analytic on } |z| \ge 1 \tag{2.3a}$$

$$\text{(ii)} \qquad v(z) + v(1/z) > 0 \text{ on } |z| = 1, \tag{2.3b}$$

then

$$\Phi(e^{i\omega}) = v(e^{i\omega}) + v(e^{-i\omega}) \tag{2.4}$$

is the spectral density of a stationary stochastic process $\{y_t; t \in \mathbb{Z}\}$ which can be represented (in uncountably many ways) by a minimal stochastic realization

$$\begin{cases} x_{t+1} = Fx_t + v_t \\ \quad y_t = h'x_t + w_t \end{cases} \tag{2.5}$$

of y, i.e. a stochastic system with $E\{x_{t+1}y_t\} = g$ obtained by passing white noise $\{v_t, w_t\}$ through a shaping filter, the transfer function of which

$$w(z) = h'(zI - F)^{-1}B + d' \tag{2.6}$$

(where B is a matrix and d a vector of appropriate dimensions) is a minimal stable spectral factor of Φ, i. e. $w(z)w(1/z)' = \Phi(z)$. In general $w(z)$ is a row vector valued rational function. If, in particular, $w(z)$ is a scalar and both its numerator and denominator polynomials stable (all zeros in the open unit disc) we say that $w(z)$ is a *minimum phase spectral factor*. All such realization (2.6) have the same *Kalman filter*

$$\hat{x}_{t+1} = F\hat{x}_t + k_t(y_t - h'\hat{x}_t); \quad \hat{x}_0 = 0 \tag{2.7}$$

(where $\hat{x}_t$ is the linear minimum-variance estimate of x_t given $\{y_0, y_1, \ldots, y_{t-1}\}$) and the gain

$$k_t \;=\; (1 - h'P_t h)^{-1}(g - FP_t h) \tag{2.8}$$

is determined by solving the corresponding matrix Riccati equation (2.2) with initial condition

$$P_0 \;=\; 0 \tag{2.9}$$

It is well-known that, under these conditions, P_t tends monotonically to the stable equilibrium of (2.2) [13, 26]. The question addressed in this paper is what happens to the solution (2.2) when the parameters have been chosen such that $v(z)$ is no longer positive real.

Without loss of generality, we shall henceforth take (F, g, h) in the observer canonical form

$$F \;=\; \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \;;\quad g \;=\; \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \;;\quad h \;=\; \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.10}$$

in terms of which we may write

$$F = J - ah'$$

where a is the column vector $(a_1, a_2, \ldots, a_n)'$ and J is the obvious shift matrix. Consequently, the Riccati equation is determined by the $2n$ parameters (a, g) and there are also the coefficients of the rational function $v(z)$, i.e.

$$v(z) = \frac{1}{2} + \frac{g_1 z^{n-1} + g_2 z^{n-2} + \cdots + g_n}{z^n + a_1 z^{n-1} + \cdots + a_n}. \tag{2.11}$$

For simplicity we shall write this

$$v(z) = \frac{1}{2} + \frac{g(z)}{a(z)} = \frac{1}{2}\frac{b(z)}{a(z)} \tag{2.12}$$

where $b(z) := a(z) + 2g(z)$ is a monic polynomial of degree n. It is easy to see that, if $v(z)$ is positive real, then

$$\text{(i)} \qquad D(z,z^{-1}) = \frac{1}{2}[a(z)b(1/z) + a(1/z)b(z)] > 0 \text{ on } |z| = 1 \tag{2.13}$$

$$\text{(ii)} \qquad a(z) \text{ has all its zeros in } |z| < 1 \tag{2.14}$$

$$\text{(iii)} \qquad b(z) \text{ has all its zeros in } |z| < 1 \tag{2.15}$$

Conversely, if (i) plus either (ii) or (iii) hold, then $v(z)$ is positive real.

To determine the Kalman filter we can, instead of the Riccati equation (2.2), use the algorithm

$$a(t+1) = \frac{1}{1-g_1(t)}[a(t) + (I-J)g(t)]; \qquad a(0) = a \tag{2.16a}$$

$$g(t+1) = \frac{1}{1-g_1(t)^2}[-g_1(t)a(t) + (J - g_1(t)I)g(t)]; \quad g(0) = g \tag{2.16b}$$

consisting of $2n$ nonlinear first-order difference equations in terms of which

$$k_t = a(t) + g(t) - a \tag{2.17}$$

This algorithm is a version, appearing in [27], of the fast Kalman filtering algorithm introduced in [25]. (Also see [5] where these matters are reviewed; in the notations of this paper $a(t) = q_t - q_t^*$.) Suppose $r_t := \prod_{k=0}^{t-1}[1-g_1(k)^2]$ and the monic polynomials $a_t(z)$ and $b_t(z) := a_t(z) + 2g_t(z)$ are formed from $a(t)$ and $b(t) := a(t) + 2g(t)$ as above, then it is shown in [27] that the equality

$$r_t[a_t(z)b_t(1/z) + a_t(1/z)b_t(z)] = 2D(z,z^{-1}) \tag{2.18}$$

is preserved along the trajectory of (2.16). It is also shown in [27] that $a_t(z)$ and $b_t(z)$ have all their zeros in the unit disc $|z| < 1$. Consequently, if $v(z)$ is positive real, then so is

$$v_t(z) = \frac{1}{2}\frac{b_t(z)}{a_t(z)} = \frac{1}{2} + \frac{g_t(z)}{a_t(z)} \tag{2.19}$$

for each $t = 1, 2, 3, \dots$, so that each $(a(t), g(t))$ is an admissible pair of parameters for the Kalman filtering problem, corresponding to stochastic systems.

3. Systems Theoretic Enhancements of Some Classical Positivity Results

One of the main results of this paper is to establish and analyze the fact that filtering algorithms do converge for parameter values which do not correspond to a *bona fide* stochastic system and, hence, which do not satisfy the relevant positivity conditions. These positivity conditions can be expressed in terms of a transfer function being positive real or a Toeplitz matrix being positive definite, as well as a number of other conditions involving familiar objects from classical analysis and systems theory. Our main result is just one manifestation of the fact that several classical and more recent results containing positivity and positive real functions have, of course, natural enhancements to statements concerning broader classes of nonsingular matrices and systems. In this section we develop this theme in the context of several particular results which we shall find to be very useful in the remaining sections.

It is well-known that a function $v(z)$ with the Laurent expansion

$$v(z) = \frac{1}{2} + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + \cdots \tag{3.1}$$

around $z=\infty$ is positive real if and only if the Toeplitz matrices

$$T_t \;=\; \begin{bmatrix} 1 & c_1 & \cdots & c_t \\ c_1 & 1 & \cdots & c_{t-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_t & c_{t-1} & \cdots & 1 \end{bmatrix} \tag{3.2}$$

are positive definite for all $t=1,2,3,\dots$.

A simpler test of positive realness due to Schur [32] can be described in terms of the Szegö polynomials $\{\varphi_0(z),\varphi_1(z),\varphi_2(z),\dots\}$, a sequence of monic polynomials

$$\varphi_t(z) \;=\; z^t+\varphi_{t1}z^{t-1}+\cdots+\varphi_{tt} \tag{3.3}$$

which are orthogonal on the unit circle. Similarly, we define the reversed polynomial $\varphi_t^*(z)$ as

$$\varphi_t^*(z)=\varphi_{tt}z^t+\varphi_{t,t-1}z^{t-1}+\cdots+1 \tag{3.4}$$

The Szegö polynomials are then determined from the sequence $\{c_1,c_2,c_3,\dots\}$ through the polynomial recursions

$$\begin{cases} \varphi_{t+1}(z)=z\varphi_t(z)-\gamma_t\varphi_t^*(z) \;\;;\quad \varphi_0(z)=1 \\ \varphi_{t+1}^*(z)=\varphi_t^*(z)-\gamma_t z\varphi_t(z) \;\;;\quad \varphi_0^*(z)=1 \end{cases} \tag{3.5}$$

where $\{\gamma_0,\gamma_1,\gamma_2,\dots\}$ are the *Schur parameters*

$$\gamma_t \;=\; \frac{1}{r_t}\sum_{k=0}^{t}\varphi_{t,t-k}c_{k+1} \tag{3.6}$$

and $\{r_0,r_1,r_2,\dots\}$ are given by the recursion

$$r_{t+1} \;=\; (1-\gamma_t^2)r_t \;\;;\quad r_0=1 \tag{3.7}$$

the algorithm terminating if $|\gamma_t|$ becomes one. Indeed, it has been shown by Schur that

$$T_t>0\Leftrightarrow|\gamma_k|<1 \quad \text{for } k=0,1,2,\dots,t-1. \tag{3.8}$$

It is also classical that the function (3.1) has an infinite *Schur parameter sequence* $\{\gamma_0,\,\gamma_1,\,\gamma_2,\,\dots\}$ if and only if $|\gamma_t|$ never becomes one – otherwise, the Schur parameter sequence is finite ending with a term of modulus one – and that for each $t=1,2,3,\dots$, there is a one-to-one correspondence between the set of all subsequences $\{c_1,c_2,\dots,c_t\}$ such that T_k is nonsingular for $k=1,2,\dots,t$ and the set of all subsequences $\{\gamma_0,\gamma_1,\dots,\gamma_{t-1}\}$ such that $|\gamma_k|\neq 1$ for $k=0,1,2,\dots,t-2$.

That these claims also hold for nonpositive data follows from the following well-known enhancement of the positive result (3.8).

Proposition 3.1. $\det T_t=\prod_{k=0}^{t}r_k$.

As a second illustration of this theme, Kimura [22] and Georgiou [16] have independently shown that to any positive real function (3.1) with the first n c-coefficients prescribed, or alternatively with $\gamma:=(\gamma_0,\gamma_1,\dots,\gamma_{n-1})'$ fixed, there is a unique vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)'$ of real numbers such that

$$v(z)=\frac{1}{2}\frac{\psi_n(z)+\alpha_1\psi_{n-1}(z)+\cdots+\alpha_n\psi_0(z)}{\varphi_n(z)+\alpha_1\varphi_{n-1}(z)+\cdots+\alpha_n\varphi_0(z)} \tag{3.9}$$

where $\{\psi_0,\psi_1,\psi_2,\dots\}$ are the Szegö polynomials obtained by exchanging the Schur parameters $\{\gamma_t\}$ by $\{-\gamma_t\}$. This is a useful parameterization for the *rational covariance extension problem* [21], but, as we shall now demonstrate, (3.9) is actually a general interpolation formula which holds regardless of whether $v(z)$ is positive real, provided that the algorithm does not terminate for $t < n$. In fact, it follows that there is a one-to-one correspondence between the open dense set in $\mathbb{R}^{2n}$ of $2n$ parameters (α, γ) for which none of the elements of the vector $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1})' \in \mathbb{R}^n$ has modulus one, and the corresponding open dense set of $(a, g) \in \mathbb{R}^{2n}$.

Theorem 3.2. *Let $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1})'$ be an arbitrary vector in $\mathbb{R}^n$ such that $\gamma_k^2 \neq 1$ for $k = 0, 1, \dots, n-2$, let $\{\varphi_k(z), \psi_k(z); k = 0, 1, \dots, n-1\}$ be the corresponding polynomials generated by* (3.5)*, and set $c_1 := \gamma_0$ and*

$$c_{k+1} := r_k \gamma_k - \sum_{j=0}^{k-1} \varphi_{k,k-j}\, c_{j+1} \tag{3.10}$$

for $k = 1, 2, \dots, n-1$, where $r_1, r_2, \dots, r_n$ are defined by (3.7)*. Let $a(z)$ and $b(z)$ be arbitrary monic polynomials of degree n such that*

$$\frac{b(z)}{2a(z)} = \frac{1}{2} + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_n z^{-n} + \cdots . \tag{3.11}$$

Then there is a unique $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)' \in \mathbb{R}^n$ such that

$$a(z) = \varphi_n(z) + \alpha_1 \varphi_{n-1}(z) + \cdots + \alpha_n \tag{3.12a}$$

$$b(z) = \psi_n(z) + \alpha_1 \psi_{n-1}(z) + \cdots + \alpha_n \tag{3.12b}$$

The proof of Theorem 3.2 is based on the following lemma.

Lemma 3.3. *Let the polynomials $\{\varphi_k(z), \psi_k(z);\, k = 0, 1, \dots, n-1\}$ and the sequence $\{c_1, c_2, \dots, c_n\}$ be as defined in Theorem 3.2. Then*

$$\Psi_{n+1} = C_{n+1} \Phi_{n+1} \tag{3.13}$$

where Φ, Ψ and C are the nonsingular $(n+1) \times (n+1)$-matrices

$$\Phi_{n+1} = \begin{bmatrix} 1 & & & & \\ \varphi_{n1} & 1 & & & \\ \varphi_{n2} & \varphi_{n-1,1} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \varphi_{nn} & \varphi_{n-1,n-1} & \varphi_{n-2,n-2} & \cdots & 1 \end{bmatrix} \tag{3.14a}$$

$$\Psi_{n+1} = \begin{bmatrix} 1 & & & & \\ \psi_{n1} & 1 & & & \\ \psi_{n2} & \psi_{n-1,1} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \psi_{nn} & \psi_{n-1,n-1} & \psi_{n-2,n-2} & \cdots & 1 \end{bmatrix} \tag{3.14b}$$

$$C_{n+1} = \begin{bmatrix} 1 & & & & \\ 2c_1 & 1 & & & \\ 2c_2 & 2c_1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 2c_n & 2c_{n-1} & 2c_{n-2} & \cdots & 1 \end{bmatrix} \tag{3.14c}$$

Proof. We want to prove that

$$\psi_{tk} = 2c_k + 2c_{k-1}\varphi_{t1} + 2c_{k-2}\varphi_{t2} + \cdots + 2c_1 \varphi_{t,k-1} + \varphi_{tk}$$

for all $t \geq k$, or equivalently

$$\rho_{tk} = c_k + c_{k-1}\varphi_{t1} + c_{k-2}\varphi_{t2} + \cdots + c_1 \varphi_{t,k-1} \tag{3.15}$$

for all $t = k, k+1, \dots, n$, where $\{\rho_{tk}\}$ are the coefficients of the polynomials

$$\rho_t(z) = \frac{1}{2}[\psi_t(z) - \varphi_t(z)] \tag{3.16}$$

Then, the recursions in φ_t and ψ_t imply that

$$\rho_{t+1}(z) = z\rho_t(z) + \gamma_t \pi_t^*(z), \tag{3.17}$$

where $\pi_t^*(z)$ is the reversed polynomial of

$$\pi_t(z) = \frac{1}{2}[\psi_t(z) + \varphi_t(z)]. \tag{3.18}$$

i.e. $\pi_t^*(z) := z^n \pi_t(1/z)$. We also recall from the literature [24], [1], [17] that the coefficients of $\{\varphi_t\}$ satisfy the *normal equations*

$$\begin{bmatrix} 1 & c_1 & \cdots & c_{t-1} \\ c_1 & 1 & \cdots & c_{t-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{t-2} & c_{t-3} & \cdots & c_1 \\ c_{t-1} & c_{t-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \varphi_{tt} \\ \varphi_{t,t-1} \\ \vdots \\ \varphi_{t1} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r_t \end{bmatrix} \tag{3.19}$$

having the Toeplitz matrix T_t as its coefficient matrix. As we have pointed out above, T_t is nonsingular if our basic assumption that $|\gamma_k| \neq 1$ for all $k = 0, 1, 2, \dots, t-1$ holds [1]. It follows from (3.5) that $\varphi_{tt} = -\gamma_{t-1}$, and consequently $\psi_{tt} = \gamma_{t-1}$, and thus $\rho_{tt} = \varphi_{tt}$. Therefore, we see from the first equation (3.19) that (3.17) holds for $t = k$. We now proceed by induction. Suppose (3.15) holds for $t = s$, where $k \leq s \leq n-1$. We want to prove that it holds for $t = s+1$. To this end, note that for $t = s$ one of the normal equations reads

$$c_k + c_{k-1}\varphi_{s1} + c_{k-2}\varphi_{s2} + \cdots + c_1\varphi_{s,k-1} + \varphi_{sk} + c_1\varphi_{s,k+1} + \cdots + c_{s-k}\varphi_{ss} = 0.$$

But, in view of the induction hypothesis, this can be written

$$\rho_{sk} + \varphi_{sk} + c_1\varphi_{s,k+1} + \cdots + c_{s-k}\varphi_{ss} = 0$$

and therefore, since $\pi_{sk} = \rho_{sk} + \varphi_{sk}$,

$$\pi_{sk} = -c_1\varphi_{s,k+1} - \cdots - c_{s-k}\varphi_{ss} \tag{3.20}$$

Now identifying coefficients in the polynomial recursion (3.18) we obtain

$$\rho_{s+1,k} = \rho_{sk} + \gamma_s \pi_{s,s+1-k}$$

which, after inserting (3.20) and applying the induction hypothesis, takes the form

$$\rho_{s+1,k} = c_k + c_{k-1}(\varphi_{s1} - \gamma_s\varphi_{ss}) + \cdots + c_1(\varphi_{s,k-1} - \gamma_s\varphi_{s,s+2-k}).$$

But it follows from (3.5) that $\varphi_{s+1,k} = \varphi_{sk} - \gamma_s\varphi_{s,s+1-k}$, and therefore (3.15) holds for $t = s+1$ as required. Hence the lemma follows by induction. □

Proof of Theorem 3.2. Since $\{\varphi_t\}$ and $\{\psi_t\}$ are families of monic polynomials of increasing degree t, there are $\alpha, \beta \in \mathbb{R}^n$ such that

$$a(z) = \varphi_n(z) + \alpha_1\varphi_{n-1}(z) + \cdots + \alpha_n,$$
$$b(z) = \psi_n(z) + \beta_1\psi_{n-1}(z) + \cdots + \beta_n.$$

Then (3.11) yields

$$\begin{aligned} &\psi_n(z) + \beta_1\psi_{n-1}(z) + \cdots + \beta_n \\ =&[\varphi_n(z) + \alpha_1\varphi_{n-1}(z) + \cdots + \alpha_n][1 + 2c_1z^{-1} + 2c_2z^{-2} + \cdots] \end{aligned}$$

Therefore, identifying coefficients of nonnegative powers of z, we have

$$\Psi_{n+1}\begin{bmatrix}1\\ \beta\end{bmatrix} = C_{n+1}\Phi_{n+1}\begin{bmatrix}1\\ \alpha\end{bmatrix}$$

which, by Lemma 3.3, implies that $\beta = \alpha$. □

Corollary 3.4. *Consider the maps $\gamma := (\gamma_0, \gamma_1, \dots, \gamma_{n-1}) \to \Phi_{n+1}(\gamma)$ and $\gamma \to \Psi_{n+1}(\gamma)$ defined through* (3.5)*, the corresponding recursion for $\{\psi_t\}$,* (3.14a) *and* (3.14b)*. Then $\Psi_{n+1}(\gamma) = \Phi_{n+1}(-\gamma)$ and $\Phi_{n+1}(0) = \Psi_{n+1}(0) = I_{n+1}$. Moreover,*

$$\begin{bmatrix}1\\ a\end{bmatrix} = \Phi_{n+1}(\gamma)\begin{bmatrix}1\\ \alpha\end{bmatrix} \qquad \begin{bmatrix}1\\ b\end{bmatrix} = \Psi_{n+1}(\gamma)\begin{bmatrix}1\\ \alpha\end{bmatrix} \tag{3.21}$$

4. The Fast Filtering Algorithm and Its Invariant Integrals

One of the principal goals of this section is to express the fast filtering algorithm (2.16) in a more convenient way in terms of the parameters (α, γ) entering in the generalization, Theorem 3.2, of the Kimura-Georgiou parameterization of positive real systems. As a preliminary step, we shall first show that this parameterization constitutes in fact a *bona fide* change of coordinates. In the language of classical algebraic geometry, the map defined by (3.12) is a birational isomorphism [33]. More explicitly, consider the set

$$U_\gamma = \{(\alpha, \gamma) \in \mathbb{R}^{2n} \mid \gamma_i^2 \neq 1, i = 0, 1, \dots, n-2\}$$

Also, by virtue of (3.11), the generalized "correlation" coefficients $c_1, c_2, \dots, c_n$ are functions of (a, b) so that we may define the open, dense set

$$V_c = \{(a, b) \in \mathbb{R}^{2n} \mid \det T_t \neq 0, i = 1, 2, \dots, n-1\}$$

We shall show that that the polynomial map $\mathcal{F}$ is a bijection of U_γ with V_c, having a rational inverse; so that $\mathcal{F}$ is indeed a birational isomorphism.

Proposition 4.1. *The map $\mathcal{F}$, defined by* (3.12)*, sending $(\alpha, \gamma) \in \mathbb{R}^{2n}$ to $(a, b) \in \mathbb{R}^{2n}$ is a polynomial map given by*

$$\begin{cases} a = \varphi_n(\gamma) + \Phi_n(\gamma)\alpha \\ b = \psi_n(\gamma) + \Psi_n(\gamma)\alpha \end{cases} \tag{4.1}$$

where $\varphi_n := (\varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nn})'$ and $\psi_n := (\psi_{n1}, \psi_{n2}, \dots, \psi_{nn})'$, and $\Phi_n(\gamma)$ and $\Psi_n(\gamma)$ are given by (3.14)*. Moreover, $\mathcal{F} : \mathcal{U}_\gamma \to \mathcal{V}$ is a bijection with a rational inverse $\mathcal{F}^{-}$.*

Proof. On V_c the map $\mathcal{F}$ has an inverse $\mathcal{F}^{-1}$ defined in the following way: (a, b) defines through (3.11) a sequence $\{c_1, c_2, \dots, c_n\}$ which, by (3.6), corresponds to a vector $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1})'$, from which in turn the polynomials $\{\varphi_k(z); k = 0, 1, 2, \dots, n-1\}$ can be defined. Then $\alpha \in \mathbb{R}^n$ is uniquely determined by (3.12a). Finally, (4.1) follows from (3.21) □

Recall from [27] or from [5] that if $\{\gamma_0, \gamma_1, \gamma_2, \dots\}$ is the (infinite or finite) Schur parameter sequence of (2.1), as defined in Section 3, then

$$\gamma_t = g_1(t) \qquad t = 0, 1, 2, \dots \tag{4.2}$$

where g_1 is generated by the fast filtering algorithm (2.16). A key observation now is that (2.16) is a time-invariant dynamical system in parameter space. In particular, let us stress the following simple but important observation.

Lemma 4.2. *Let $v(z)$ be defined as in Section 2 and let $\{\gamma_0, \gamma_1, \gamma_2, \dots\}$ be its (infinite of finite) Schur parameter sequence. Then, for each $t = 0, 1, 2, \dots,$ as long as the algorithm* (2.16) *does not escape, $v_t(z)$ defined by* (2.19) *has the Schur parameter sequence $\{\gamma_t, \gamma_{t+1}, \gamma_{t+2}, \dots\}$.*

Proof. The fast filtering algorithm (2.16) is a time-invariant dynamical system, and unless it has escaped it will therefore trivially generate via (4.2) the sequence $\{\gamma_t, \gamma_{t+1}, \gamma_{t+2}, \dots\}$ if initialized at $(a(t), g(t))$ corresponding to $v_t(z)$. □

Corollary 3.4 allows us to change coordinates in the fast algorithm, expressing it instead in terms of (α, γ), where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \qquad \gamma = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{bmatrix} \tag{4.3}$$

as long as $(\alpha, \gamma) \in U_\gamma$, i.e. as long as $\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$ is the initial subsequence of a Schur parameter sequence.

Theorem 4.3. *Let the rational function $v(z)$ defined by* (2.1) *have a Schur parameter sequence such that $\gamma_k^2 \neq 1$ for $k = 0, 1, \dots, n-1$. Then the fast filtering algorithm takes the form*

$$\alpha(t+1) = A(\gamma(t))\alpha(t), \qquad \alpha(0) = \alpha \tag{4.4a}$$
$$\gamma(t+1) = G(\alpha(t+1))\gamma(t), \qquad \gamma(0) = \gamma \tag{4.4b}$$

in the coordinates of the generalized Kimura-Georgiou parameterization, where the maps $A, G : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ are defined as

$$A(\gamma) = \begin{bmatrix} \frac{1}{1-\gamma_{n-1}^2} & \frac{\gamma_{n-1}\gamma_{n-2}}{(1-\gamma_{n-1}^2)(1-\gamma_{n-2}^2)} & \cdots & \frac{\gamma_{n-1}\gamma_0}{(1-\gamma_{n-1}^2)\cdots(1-\gamma_0^2)} \\ 0 & \frac{1}{1-\gamma_{n-2}^2} & \cdots & \frac{\gamma_{n-2}\gamma_0}{(1-\gamma_{n-2}^2)\cdots(1-\gamma_0^2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{1-\gamma_0^2} \end{bmatrix} \tag{4.5}$$

and

$$G(\alpha) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{bmatrix} \tag{4.6}$$

More precisely, if (α, γ) are the parameters of $v(z)$ in the representation (3.9)*, then $(\alpha(t), \gamma(t))$ are the parameters of $v_t(z)$, as defined in* (2.19)*, for each point in the finite or infinite orbit of (α, γ). Moreover, if $\{\gamma_0, \gamma_1, \gamma_2, \dots\}$ is the sequence of Schur parameters of $v(z)$, then the sequence $\{\gamma_t, \gamma_{t+1}, \gamma_{t+2}, \dots\}$ obtained by deleting the first t elements is the Schur parameters sequence of $v_t(z)$. In fact,*

$$\gamma_k(t) = \gamma_{t+k} \tag{4.7}$$

and therefore the Schur parameters are updated according to the recursion

$$\gamma_{t+n} = -\alpha_1(t+1)\gamma_{t+n-1} - \alpha_2(t+1)\gamma_{t+n-2} - \cdots - \alpha_n(t+1)\gamma_t \tag{4.8}$$

Finally, the gain sequence $\{k_0, k_1, k_2, \dots,\}$ *of the Kalman filter is given by*

$$k_t = \Pi_n(\gamma(t))\alpha(t) + \pi_n(\gamma(t)) - \Phi_n(\gamma)\alpha - \varphi_n(\gamma) \tag{4.9}$$

where $\varphi_n(\gamma)$ *and* $\psi_n(\gamma)$ *are* n*-vectors of coefficients of* $\varphi(z)$ *and* $\psi(z)$*, and*

$$\Pi_n(\gamma) = \frac{1}{2}[\Phi_n(\gamma) + \Psi_n(\gamma)],$$
$$\pi_n(\gamma) = \frac{1}{2}[\varphi_n(\gamma) + \psi_n(\gamma)].$$

For a proof, we refer the reader to the Appendix.

In Section 5 we shall show that the dynamical system (4.4) evolves on an invariant manifold X_D defined by the preserved pseudo-polynomial (2.18), which we write in the form

$$D(z, z^{-1}) = d(z) + d(1/z) \tag{4.10a}$$

where

$$d(z) = \frac{1}{2}d_0 + d_1 z + d_2 z^2 + \cdots + d_n z^n. \tag{4.10b}$$

The symmetric pseudo-polynomial D is determined by the initial condition (α, γ) in a manner described by the following lemma, the proof of which is given in the Appendix.

Lemma 4.4. *Let* $D(z, z^{-1})$ *be the pseudo-polynomial* (4.10) *corresponding to the initial condition* (α, γ)*. Then*

$$d_0 = \alpha_n^2 + r_1\alpha_{n-1}^2 + \cdots + r_n, \tag{4.11}$$

where $r_1, r_2, \dots, r_n$ *are defined by* (3.7)*, and* $d_i := d_i^{(n)}(\alpha, \gamma)$ *for* $i = 1, 2, \dots, n$*, where* $d_i^{(n)}$ *is determined recursively by*

$$\begin{aligned}
&d_1^{(1)}(\alpha_1, \gamma_0) = \alpha_1;\\
&d_i^{(k)}(\alpha_1, \dots, \alpha_k, \gamma_0, \dots, \gamma_{k-1}) = (1 - \gamma_0^2)d_i^{(k-1)}(\alpha_1, \dots, \alpha_{k-1}, \gamma_1, \dots, \gamma_{k-1})\\
&\qquad + \alpha_k \sum_{j=1}^{k} \alpha_{k-j}\pi_{j,j-i}, \qquad \text{for } i = 1, 2, \dots, k-1;\\
&d_k^{(k)}(\alpha_1, \dots, \alpha_k, \gamma_0, \dots, \gamma_{k-1}) = \alpha_k;
\end{aligned}$$

where $\{\pi_{jl}\}$ *are the coefficients of the polynomials*

$$\pi_j(z) = z^j + \pi_{j1}z^{j-1} + \cdots + \pi_{jj}$$

generated by the polynomial recursion

$$\begin{cases} \pi_{t+1}(z) = (1+z)\pi_t(z) + (\gamma_t\gamma_{t-1} - 1)z\pi_{t-1}(z); \\ \pi_0 = 1, \quad \pi_1(z) = z \end{cases} \tag{4.12}$$

and $\pi_{ji} = 0$ *for* $i > j$*. Moreover, if* $\gamma_k^2 \neq 1$ *for* $k = 0, 1, \dots, n-1$*, then at least one of the coefficients* $d_0, d_1, \dots, d_n$ *of the pseudo-polynomial* $D(z, z^{-1})$ *is nonzero.*

Comparing coefficients of $(z^i + z^{-i})$ in (2.18) we see that

$$r_t d_i(\alpha(t), \gamma(t)) = d_i(\alpha(0), \gamma(0)) \quad i = 0, 1, 2, \dots, n \tag{4.13}$$

for all $t \in \mathbb{Z}$ along the trajectory of the dynamical system (4.4). Hence the $n+1$ functions $d_i(\alpha, \gamma)$, $i = 0, 1, \dots, n-1$ defined in Lemma 4.4, are invariant under the evolution of (4.4) up to a (common) scaling factor; i.e. these $(n+1)$ functions are projectively invariant. We can obtain n invariant quantities, either by viewing the pseudo-polynomial, in terms of homogeneous coordinates, as a point in $\mathbb{P}^n$ (see [33]), or equivalently by dividing each of the $(n+1)$ equations in (4.13) by any one of the $(n+1)$ functions which is nonzero (by Lemma 4.4, there is always one), obtaining rational functions having values independent of r_t and hence depending only on (α, γ). That is, we can view the pseudo-polynomial D either as determining $(n+1)$ projectively invariant functions $T_1, \dots T_{n+1}$ or as determining a map $\overline{T}$ to $\mathbb{P}^n$:

$$\begin{array}{ccc} \mathbb{R}^{2n} & \xrightarrow{T} & \mathbb{R}^{n+1} - \{0\} \\ & \overline{T} \searrow & \downarrow \Pi \\ & & \mathbb{P}^n \end{array}$$

where $T = (T_1, \dots, T_{n+1})$ and $\overline{T} = \Pi \circ T$ where

$$\Pi(x_1, \dots, x_{n+1}) = [x_1, \dots, x_{n+1}].$$

In this way one might expect (2.18) to define an n-fold X_D in $\mathbb{R}^{2n}$. Indeed, this analytic set will be a smooth n-manifold at a point (α, γ) provided Jac $T\big|_{(\alpha,\gamma)}$ has an n-dimensional kernel. We shall return to this question in Section 5 and Section 6 after having introduced some additional analytic tools.

¿From Theorem 4.3 and Lemma 4.4 it is clear that the fast filtering algorithm has a quasi-nested structure in the sense that whenever $\alpha_n = \alpha_{n-1} = \dots = \alpha_{k+1} = 0$ but $\alpha_k \neq 0$, the dynamical system (4.4) and the invariant set X_D reduce to the k-dimensional case with the Schur parameter sequence shifted $n-k$ steps. (This is related to the occurrence of invariant directions [3] in the corresponding matrix Riccati equation (2.2) as pointed out in [27] and further elaborated on [31].) As explained in [25, 27, 26, 5] the fast algorithm is intimately connected to the Szegö orthogonal polynomial recursion (3.5), which in fact was the basic tool in the original derivation [25]. In view of this and the analysis above one would expect that it would also be connected to the Schur algorithm. Indeed, this has been shown in a recent paper [8].

5. **Invariant Manifolds and Local Convergence for the Fast Filtering Algorithms**

We now turn to a stability analysis of the equilibria of the fast filtering algorithms, expressed in the form (4.4), i.e.

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}_{t+1} = f \begin{pmatrix} \alpha_t \\ \gamma_t \end{pmatrix} \tag{5.1a}$$

where

$$f \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{bmatrix} A(\gamma)\alpha \\ G(A(\gamma)\alpha)\gamma \end{bmatrix} \tag{5.1b}$$

For the stability analysis of the fast filtering algorithms, we shall need the geometric concepts of stable, unstable and center manifolds, which play a role for nonlinear systems similar to the role played by generalized eigenspaces for the stability analysis of linear systems. Because this role is so important in determining stability, especially in the critical case, we precede our analysis of the local stability of the fast filtering algorithms with an introductory discussion of local invariant manifolds for nonlinear systems. As supplementary references we recommend, among other texts, Guckenheimer and Holmes [18], and Marsden and McCracken [28].

At an equilibrium $(\alpha_\infty, \gamma_\infty)$ of (5.1)

$$\begin{bmatrix} \alpha - \alpha_\infty \\ \gamma - \gamma_\infty \end{bmatrix}_{t+1} = C \begin{bmatrix} \alpha - \alpha_\infty \\ \gamma - \gamma_\infty \end{bmatrix}_t + O(\|\alpha - \alpha_\infty\|^2 + \|\gamma - \gamma_\infty\|^2)$$

determines to first order a linear system

$$\begin{bmatrix} \bar{\alpha} \\ \bar{\gamma} \end{bmatrix}_{t+1} = C \begin{bmatrix} \bar{\alpha} \\ \bar{\gamma} \end{bmatrix}_{t},$$

where $\bar{\alpha} = \alpha - \alpha_\infty$, $\bar{\gamma} = \gamma - \gamma_\infty$. Denote by s the number of eigenvalues of the matrix C having modulus less than one, counting roots of the characteristic polynomial with their algebraic multiplicities. Similarly, denote by u the number of eigenvalues having modulus greater than one and by c the number of eigenvalues having modulus one. It is well known that if $u \geq 1$ then (5.1) is unstable, so we shall suppose for the moment that $u = 0$. In this case, if $c = 0$, then $(\alpha_\infty, \gamma_\infty)$ is an asymptotically stable equilibrium for the system (5.1), with all solutions converging geometrically to the equilibrium. The critical case, $c \neq 0$, is more subtle, even for linear systems where Lyapunov stability is determined by the geometric multiplicities of the eigenvalues lying on the unit circle.

Remarkably, the linear case can in fact be analyzed geometrically in a manner which can be adapted to the critical nonlinear case, *mutatis mutandis.* Denote by V^s the sum of the generalized eigenspaces corresponding to eigenvalues inside the unit disk, by V^u the sum of the generalized eigenspaces corresponding to eigenvalues outside the unit disk, and by V^c the sum of the generalized eigenspaces corresponding to eigenvalues lying on the unit circle. Then, we have

$$\dim V^s = s, \quad \dim V^u = u, \quad \text{and } \dim V^c = c.$$

In particular, there is a direct sum decomposition of the state space consisting of three invariant subspaces

$$\mathbb{R}^{2n} = V^s \oplus V^u \oplus V^c$$

Moreover, the evolution of the entire linear system is a superposition of the three motions on the constituent invariant subspaces: the asymptotically stable motion on V^s, the asymptotically expanding motion on V^u, and the motion on V^c which is determined by the dimension of the Jordan blocks corresponding to the eigenvalues of unit modulus. For example, if $u = 0$ as assumed above, it can be easily verified that any trajectory of the full linear system converges geometrically to a trajectory lying on V^c. Therefore, if $u = 0$ the (asymptotic) stability or instability of the full linear system is determined by the (asymptotic) stability or instability of the reduced dynamics on V^c.

In the nonlinear case, the geometric situation is similar. It is now classical that the nonlinear analogue of V^s can be locally defined as the set W^s of initial conditions which converge to the equilibrium at a geometric rate. The set W^s, referred to as the *stable manifold*, is known to be locally invariant and to be locally a smooth submanifold of the state space, having dimension s. A similar characterization of the set of geometrically expanding points can be given, leading to the unstable manifold W^u, which is locally defined as an invariant, smooth submanifold of dimension u. In this context, it is easy to see that if $u = 0$ and if $c = 0$ then W^s is a neighborhood of the equilibrium and therefore the equilibrium is locally asymptotically stable. An analysis of the critical case, $u = 0$ but $c \neq 0$ is facilitated by the existence of a center manifold, W^c, which plays a role analogous to the role played for linear systems by V^c. The existence of a center manifold has been established only relatively recently in part due to the absence of an explicit characterization of W^c as a set, a fact which also partially explains the fact that center manifolds need not be unique. This existence result is only part of the fundamental "center manifold theorem", which we now describe in more detail.

Center Manifold Theorem ([18, 28])

(i) *Existence.* Suppose (5.1) is a C^{k+1} system with an equilibrium $(\alpha_\infty, \gamma_\infty)$ for which $\dim V^c = c$. Then, in a sufficiently small neighborhood of the equilibrium there exists a C^k-submanifold W^c of dimension c, which is locally invariant and for which the tangent space to W^c at $(\alpha_\infty, \gamma_\infty)$ is V^c.

(ii) *Principle of Asymptotic Phase.* Suppose further that $u = 0$ for the equilibrium $(\alpha_\infty, \gamma_\infty)$. Then, for each initial condition sufficiently close to the equilibrium there is an initial condition on W^c for which the error between the corresponding trajectories asymptotically decreases geometrically.

We shall use this theorem for convergence analysis of the fast filtering algorithm (5.1).

Lemma 5.1. *The point* (α, γ) *is an equilibrium of the fast filtering algorithm* (5.1) *if and only if* $\gamma = 0$. *The Jacobian of* f *at the equilibrium* $(\alpha, 0)$ *is given by*

$$\text{Jac } f\Big|_{(\alpha,0)} = \begin{bmatrix} I & 0 \\ 0 & G(\alpha) \end{bmatrix}$$

where $G(\alpha)$ *is defined as in* (4.6).

Proof. Since $A(0) = I$, $(\alpha, 0)$ is clearly an equilibrium for each $\alpha \in \mathbb{R}^n$. It remains to show that each equilibrium is of this form. To this end, let (α, γ) satisfy

$$\alpha = A(\gamma)\alpha \tag{5.2a}$$

$$\gamma = G(\alpha)\gamma \tag{5.2b}$$

The last of equations (5.2a) reads

$$\alpha_n = \frac{\alpha_n}{1 - \gamma_0^2}$$

which requires that either α_n or γ_0 is zero, i.e. $\alpha_n \gamma_0 = 0$. In view of this, the second last equation becomes

$$\alpha_{n-1} = \frac{\alpha_{n-1}}{1 - \gamma_1^2}$$

which implies that $\alpha_{n-1}\gamma_1 = 0$. Proceeding in this manner we see that

$$\alpha_{n-i}\gamma_i = 0; \quad i = 0, 1, 2, \dots, n-1$$

and therefore the last of equations (5.2b) yields $\gamma_{n-1} = 0$. Then solving (5.2b) successively from the bottom yields $\gamma = 0$ as required. Since $A(0) = I$ and $\frac{\partial A}{\partial \gamma}(0) = 0$, the Jacobian of f is as stated in the lemma. ☐

In particular, this lemma shows that whenever α corresponds to a Schur polynomial,

$$\alpha(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n, \tag{5.3}$$

the stable manifold of the fast filtering algorithm at $(\alpha, 0)$ is n-dimensional. The next result significantly refines this observation. In particular, we will characterize the stable manifold explicitly. As a preliminary, we denote by V_λ the generalized (complex) eigenspace of $G(\alpha)$ corresponding to an eigenvalue λ of $G(\alpha)$; i.e. a root of (5.3), and we define

$$s(\alpha) = \dim_{\mathbb{C}} \sum_{|\lambda|<1} V_\lambda$$

$$u(\alpha) = \dim_{\mathbb{C}} \sum_{|\lambda|>1} V_\lambda$$

$$c(\alpha) = \dim_{\mathbb{C}} \sum_{|\lambda|=1} V_\lambda$$

In particular, $(\alpha, 0)$ is hyperbolic if, and only if, $s(\alpha) + u(\alpha) = n$ or, equivalently, $c(\alpha) = 0$.

Theorem 5.2. *The dimensions of the stable manifold and unstable manifold at* $(\alpha_\infty, 0)$ *are* $s(\alpha_\infty)$ *and* $u(\alpha_\infty)$*, respectively. The dimension of a center manifold is always* $n + c(\alpha_\infty)$*. In fact, any center manifold contains an open neighborhood of* $(\alpha_\infty, 0)$ *in the* n*-dimensional equilibrium manifold*

$$E = \{(\alpha, 0) : \alpha \in \mathbb{R}^n\}$$

Moreover, if $c(\alpha_\infty) = 0$ *then the center manifold is unique and locally coincides with* E*. In this case, the equilibrium* $(\alpha_\infty, 0)$ *is Lyapunov stable if, and only if,* $u(\alpha_\infty) = 0$*, in which case the stable manifold is* n*-dimensional.*

Proof. Since any center manifold M must contain all local attractors in some neighborhood U of $(\alpha_\infty, 0)$, $M \cap U \supset E \cap U$. If $c(\alpha_\infty) = 0$ then by a dimension argument $M \cap U = E \cap U$ and hence $M \cap U$ is unique. In this case, by the center manifold theorem, the overall system will be Lyapunov stable when $u(\alpha_\infty) = 0$ and trajectories initialized at points (α, γ) sufficiently close to $E \cap A$, where $A = \{(\alpha, 0) : (5.3) \text{ is a Schur polynomial}\}$, will approach $(\alpha_\infty, 0)$ determined by (4.13) with a convergence rate

$$|\alpha_t| \, < \, C \cdot \mu^t \left\| \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \right\|$$

where $\mu = \max_{|\lambda|<1} |\lambda|$, λ an eigenvalue of $G(\alpha_\infty)$. □

Finally, suppose $(\alpha_\infty, 0)$ is an equilibrium corresponding to a Schur polynomial (5.3) so that $(\alpha_\infty, 0)$ has an n–dimensional stable manifold, $W^s(\alpha_\infty, 0)$. Let (α, γ) be an initial condition lying on $W^s(\alpha_\infty, 0)$. We have noted that the equality (2.18) will hold on the orbit $\{(\alpha_t, \gamma_t); t = 0, 1, \dots\}$ and hence must hold for $(\alpha_\infty, 0)$. From this observation we can obtain the n-invariants (4.13) in a simple form, by computing the right-hand side of (2.18) in the limit as a solution of a spectral factorization problem, namely,

$$2r_\infty \alpha_\infty(z) \alpha_\infty(1/z) = 2D(z, z^{-1}) \tag{5.4}$$

where r_∞ is the limit of r_t as $t \to \infty$.

Corollary 5.3. [5] *A necessary condition for an initial condition* (α, γ) *to generate a convergent trajectory is that the pseudo-polynomial* $D(z, z^{-1})$ *in* (2.18) *be sign definite.*

If the invariant set X_D introduced in Section 4 contains an equilibrium point, then $d_0 \neq 0$ in Lemma 4.4, and we may describe X_D in terms of the functions $\mathbb{R}^{2n} \to \mathbb{R}^n$

$$h_i(\alpha, \gamma) = 2\frac{d_i(\alpha, \gamma)}{d_0(\alpha, \gamma)} \quad i = 1, 2, \dots, n \tag{5.5}$$

where $d_i(\alpha, \gamma)$, $i = 0, 1, 2, \dots, n$, are as defined in Lemma 4.4.

Theorem 5.4. *Suppose that* (α, γ) *generates a convergent trajectory for the dynamical system* (4.4)*, and let* $D(z, z^{-1})$ *be the corresponding pseudo-polynomial* (4.10)*. Then, at each point of the trajectory,*

$$h_i(\alpha(t), \gamma(t)) = \kappa_i \quad i = 1, 2, \dots, n \tag{5.6}$$

where $\kappa_1, \kappa_2, \dots, \kappa_n$ *are constants which can be determined from the initial condition* (α, γ)*. In fact, if* $\alpha_n \neq 0$*, then* $\kappa_n \neq 0$ *and*

$$d(z) = \alpha_n[z^n + \frac{\kappa_{n-1}}{\kappa_n} z^{n-1} + \dots + \frac{1}{\kappa_n}], \tag{5.7}$$

and, if $\alpha_n = \dots = \alpha_{k+1} = 0$ *but* $\alpha_k \neq 0$*, then* $\kappa_n = \dots = \kappa_{k+1} = 0$*,* $\kappa_k \neq 0$ *and*

$$d(z) = r_{n-k} \alpha_k [z^k + \frac{\kappa_{k-1}}{\kappa_k} z^{k-1} + \dots + \frac{1}{\kappa_k}]. \tag{5.8}$$

Conversely, any point (α, γ) *such that*

$$h_i(\alpha, \gamma) = \kappa_i \quad i = 1, 2, \dots, n \tag{5.9}$$

has a (finite or infinite) orbit satisfying (5.6) *and the same pseudo-polynomial* (4.10) *modulo multiplication by a nonzero constant.*

Proof. According to Lemma 5.1, the equilibrium has the form $(\alpha_\infty, 0)$, and, since there is no finite escape, $r_t \neq 0$ for all $t \in \mathbb{Z}$. Consequently, in view of (A-16) and (4.11),

$$d_0(\alpha(t), \gamma(t)) = \frac{r_\infty}{r_t} d_0(\alpha_\infty, 0) \tag{5.10}$$

where, by (4.11),

$$d_0(\alpha, 0) = \alpha_n^2 + \alpha_{n-1}^2 + \dots + \alpha_1^2 + 1$$

is nonzero. Moreover, $r_\infty \neq 0$. In fact, if $r_\infty = 0$, (5.4) implies that $D(z, z^{-1}) \equiv 0$ which contradicts Lemma 4.4. Hence (5.10) is nonzero and the rational functions (5.5) are finite on the whole trajectory. Moreover, for all $t \in \mathbb{Z}$,

$$h_i(\alpha(t), \gamma(t)) = h_i(\alpha(0), \gamma(0)) \quad i = 0, 1, 2, \dots, n.$$

Setting $\kappa_i := h_i(\alpha(0), \gamma(0))$, $i = 1, 2, \dots, n$, we obtain (5.6). Next, note that $d_i = \frac{1}{2} d_0 \kappa_i$ and $d_n = \alpha_n$. Therefore, if $\alpha_n \neq 0$, then $\kappa_n \neq 0$ and $\frac{1}{2} d_0 = \frac{\alpha_n}{\kappa_n}$, and consequently (5.7) follows. If $\alpha_n = \dots = \alpha_{k+1} = 0$ but $\alpha_k \neq 0$, then, by Lemma 4.4, $d_i = 0$ for $i = k+1, \dots, n$ and $d_k = r_{n-k} \alpha_k \neq 0$. Hence, $\kappa_n = \dots = \kappa_{k+1} = 0$, $\kappa_k \neq 0$ and $\frac{1}{2} d_0 = \frac{r_{n-k} \alpha_k}{\kappa_k}$, and therefore (5.8) follows. Consequently, any $(\alpha, \gamma) \in \mathbb{R}^{2n}$ satisfying (5.9) has a pseudo-polynomial which differs from $D(z, z^{-1})$ by at most the nonzero constant, α_n or $r_{n-k} \alpha_k$, whichever case applies, and therefore the points on its orbit satisfy (5.6). □

In view of Theorem 5.4 it is reasonable to let the invariant set (5.9) be denoted X_D, with D interpreted as a point in projective space $\mathbb{P}^n$, as explained at the end of Section 4. We would like to determine at what points (α, γ) the invariant set X_D is an n-fold, i.e. for which (α, γ) the tangent space $T_{(\alpha, \gamma)} X_D$, or, which is the same, the kernel of the Jacobian of f at (α, γ), has dimension n.

To investigate this point, let us return to the pseudo-polynomial relation (2.18) defining the integrals (5.6). To this end, let

$$S(a)v = a(z)v(1/z) + a(1/z)v(z) \tag{5.11}$$

define an operator $S(a) : V_n \to \mathcal{D}_n$ from the vector space V_n of polynomials having degree less than or equal to n into the vector space $\mathcal{D}_n$ of symmetric pseudo-polynomials of degree at most n. Such an operator can be defined for each polynomial $a(z)$ of degree n. Relation (2.13) defining $D(z, z^{-1})$ in terms of $a(z)$ and $b(z)$ can then be written

$$S(a)b = 2D \tag{5.12}$$

and we may ask under what conditions this linear system may be solved for b in terms of D and a. It is well-known that the answer to this question depends on the location of the zeros of $a(z)$. We shall say that $(\lambda, 1/\lambda)$ is a *pair of reciprocal roots* (of multiplicity μ) of a pseudo-polynomial $D(z, z^{-1})$ provided that both λ and $1/\lambda$ are roots (of multiplicity μ). According to this definition *each* root (of multiplicity μ) at $\lambda = 1$ or $\lambda = -1$ determines a pair, $(1, 1)$ or $(-1, -1)$, of reciprocal roots (of multiplicity μ).

Lemma 5.5. *Let* ρ *be the number of reciprocal roots of* $a(z)$ *counted with multiplicity. Then*

$$\dim \ker S(a) = \rho. \tag{5.13}$$

Proof. This follows easily from the unit circle version of Orlando's formula [14]. Also see [10], noting that the Jury matrix of $a(z)$ is a matrix representation of $S(a)$. □

We may now write the invariance relation (2.18) in the form

$$r_t S(a_t) b_t = 2D. \tag{5.14}$$

The next lemma establishes notation for the subsequent analysis. Denote by H_n the hyperplane in V_n of monic polynomials. We note that for $\beta \in H_n$ the tangent space $T_\beta H_n$ to H_n at β is canonically isomorphic with V_{n-1}.

Lemma 5.6. *Let (α, γ) be a point in the invariant algebraic set X_D, defined by (5.9) in Theorem 5.4, with the property that $\gamma_k^2 \neq 1$ for $k = 0, 1, \ldots, n-1$, and let (a, b) be given by (3.12). Then the tangent space $T_{(\alpha,\gamma)} X_D$ of X_D at (α, γ) has the same dimension as the tangent space of*

$$\phi(a, b, r) = 2D \tag{5.15}$$

at $(a, b, 1)$, where $\phi : H_n \times H_n \times \mathbb{R} \to \mathcal{D}_n$ is defined by

$$\phi(a, b, r) = rS(a)b. \tag{5.16}$$

Proof. The lemma follows immediately from the fact that (5.9) is obtained from (5.15) by merely eliminating the variable r, which is nonzero since all $\gamma_k^2 \neq 1$, and changing coordinates under the bijection $\mathcal{F}$ of Corollary 3.4. □

It is not hard to characterize those tangent vectors which are annihilated by Jacobian of ϕ at $(a, b, 1)$ and hence span the tangent space of (5.15) at $(a, b, 1)$.

Lemma 5.7. *At any point $(a, b, 1)$, the kernel of the Jacobian of ϕ consists of those tangent vectors $(u, v, q_0) \in V_{n-1} \times V_{n-1} \times \mathbb{R}$ satisfying*

$$S(a)q + S(b)p = 0 \tag{5.17}$$

where

$$p(z) := u(z), \qquad q(z) := q_0 b(z) + v(z) \tag{5.18}$$

In other words, the kernel of the Jacobian of ϕ can be identified with pairs $(p, q) \in V_{n-1} \times V_n$, i.e.those polynomials of the form

$$\begin{cases} p(z) = p_1 z^{n-1} + \cdots + p_n \\ q(z) = q_0 z^n + q_1 z^{n-1} + \cdots + q_n \end{cases} \tag{5.19}$$

which satisfy the "variational equation"

$$a(z)q(1/z) + a(1/z)q(z) + b(z)p(1/z) + b(1/z)p(z) = 0 \tag{5.17$'$}$$

Proof. Consider the tangent vector $(a + \varepsilon u, b + \varepsilon v, 1 + \varepsilon q_0)$ at the point $(a, b, 1)$ where $u \in V_{n-1}, v \in V_{n-1}$ and $q_0 \in \mathbb{R}$. We compute the directional derivative of ϕ in the direction (u, v, q_0) as the limit of a Newton quotient

$$\frac{1}{\varepsilon}[\phi(a + \varepsilon u, b + \varepsilon v, 1 + \varepsilon q_0) - \phi(a, b)] \tag{5.20}$$

as $\varepsilon \to 0$, yielding (5.17$'$). □

Lemma 5.8. *Suppose α_∞ corresponds to a polynomial $\alpha(z)$, via (5.3), which has no pair of reciprocal roots. Then the invariant algebraic set X_D is a smooth submanifold of dimension n at the equilibrium $(\alpha_\infty, 0)$.*

Proof. When $\gamma = 0$ we have $a(z) = b(z) = \alpha(z)$, so that the variational equation reduces to

$$S(a)[p+q] = 0. \tag{5.21}$$

Since $a(z)$ has no pair of reciprocal roots, by Lemma 5.5, $\ker S(a) = 0$ and therefore we must have

$$p(z) = -q(z). \tag{5.22}$$

Note, in particular, that $q_0 = 0$, i. e. the tangent vectors belong to the $2n$-dimensional space with coordinates (a, b) or (α, γ) as in Corollary 3.4. Since (5.22) defines a subspace of tangent vectors having dimension n, by Lemma 5.6, (5.9) locally defines a smooth submanifold in a neighborhood of $(\alpha_\infty, 0)$ by the implicit function theorem. □

Theorem 5.9. *Let $(\alpha_\infty, 0)$ be an equilibrium. If $c(\alpha_\infty) = u(\alpha_\infty) = 0$, then the stable manifold through $(\alpha_\infty, 0)$ coincides with an open subset of the invariant n-fold X_D determined from (5.9). Moreover, any point (α, γ) on X_D corresponding to a positive real function $v(z) := \frac{1}{2}\frac{b(z)}{a(z)}$ will lie on this stable manifold and the minimum phase spectral factor of the spectral density $v(z) + v(1/z)$ will be*

$$w(z) = r_\infty^{1/2}\frac{\alpha_\infty(z)}{a(z)}. \tag{5.23}$$

Proof. Since $\alpha_\infty(z)$ is a Schur polynomial, (5.9) locally defines a smooth submanifold at $(\alpha_\infty, 0)$ by Lemma 5.8, with tangent space given by (5.22). We claim that (5.22) also characterizes, in the (a, b)-coordinates, those tangent vectors (p, q) which are vertical in the (α, γ)-coordinates at point $(\alpha_\infty, 0)$. In fact, the map $\mathfrak{F}$ of Corollary 3.4 sends the "vertical vector" $(0, \gamma)$ to

$$(a, b) = (\varphi_n(\gamma), \psi_n(\gamma)) \tag{5.24}$$

where here φ_n and ψ_n are the n-vectors of coefficients in the Szegö polynomials $\varphi_n(z)$ and $\psi_n(z)$ as functions of γ. The vertical vectors $(0, \gamma)$ at the point $(\alpha_\infty, 0)$ corresponds to the tangent vectors (p, q) of (5.24), i.e. the vectors of the form $(\frac{\partial \varphi_n}{\partial \gamma_i}(0), \frac{\partial \psi_n}{\partial \gamma_i}(0))$, $i = 1, 2, ..., n$. But, according to Corollary 3.4, $\varphi_n(\gamma) = \psi_n(-\gamma)$ so that $\frac{\partial \varphi_n}{\partial \gamma_i}(0) = -\frac{\partial \psi_n}{\partial \gamma_i}(0)$, and hence $p = -q$ as claimed.

Now recall that the vertical vectors at $(\alpha_\infty, 0)$ are precisely the vectors lying in the sum of the generalized eigenspaces for the Jacobian, corresponding to asymptotically stable eigenvalues, i.e. in the tangent space to the stable manifold at $(\alpha_\infty, 0)$. In summary, the invariant set X_D is an n-dimensional smooth submanifold near the equilibrium $(\alpha_\infty, 0)$ which it contains, provided $(\alpha_\infty, 0)$ is an asymptotically stable equilibrium. In this case, the tangent space to this submanifold at the equilibrium coincides with the tangent space to the stable manifold $W^s(\alpha_\infty, 0)$. In particular, an initial condition lying on X_D corresponding to positive real functions (2.1) will converge geometrically to $(\alpha_\infty, 0)$, in harmony with classically known convergence properties of the Kalman filter. By uniqueness of $W^s(\alpha_\infty, 0)$ we see that it coincides, in a neighborhood of $(\alpha_\infty, 0)$, with the invariant set defined by (5.9). Finally, from (2.13) and (5.4), we see that the minimum phase spectral factor $w(z)$ corresponding to $v(z)$ is given by (5.23). □

Remark 5.10. As a consequence of Theorem 5.9, we can see that the set of rational integral invariants $\{h_1, h_2, \ldots, h_n\}$, defined in (5.5), is complete. That is, there is no analytic (or meromorphic) invariant function h for which the differential dh is linearly independent of the differentials dh_i, $i = 1, 2, \ldots, n$, at some point (α, γ). Indeed, if for some point $(\bar{\alpha}, \bar{\gamma})$

$$dh(\bar{\alpha}, \bar{\gamma}) \notin \operatorname{span}\{dh_i(\bar{\alpha}, \bar{\gamma})\}_{i=1}^n$$

then for all (α, γ) in some open dense set U we must have

$$dh(\alpha, \gamma) \notin \text{span}\,\{dh_i(\alpha, \gamma)\}_{i=1}^n \tag{5.25}$$

by a standard analyticity argument. Now consider the region $\mathcal{P}_n$ of all $(\alpha, \gamma) \in \mathbb{R}^{2n}$ satisfying the positive real conditions (2.13)-(2.15) with $a(z)$ and $b(z)$ given by (4.1). For any initial condition $(\alpha(0), \gamma(0)) \in \mathcal{P}_n$, we must have

$$h(\alpha(t), \gamma(t)) = h(\alpha_\infty, 0)$$

so that, on $\mathcal{P}_n$, h is determined by its restriction $\bar{h}$ to the equilibrium set

$$E_s = \{(\alpha_\infty, 0) \in \mathbb{R}^{2n} | \alpha_\infty(z) \text{ a Schur polynomial}\}$$

On the other hand, α_∞ can be computed from $(\alpha(0), \gamma(0))$ as a rational function of the h_i. More explicitly, $h_i(\alpha(0), \gamma(0))$ determine, up to a scalar multiple, the pseudo-polynomial $D(z, z^{-1})$ via (5.4) and (A-16). From $D(z, z^{-1})$, which is positive on the unit circle, we determine (independently of the scalar multiple) the stable polynomial $\alpha_\infty(z)$, and hence α_∞, via (5.4). Therefore, on all of $\mathcal{P}_n$ we have

$$h(\alpha(0), \gamma(0)) = \bar{h}(\alpha_\infty, 0) = \bar{h}(F(h_1(\alpha(0), \gamma(0)), \ldots, h_n(\alpha(0), \gamma(0)), 0) \tag{5.26}$$

where F is the rational function defined by (5.4), (4.10), and (5.4). In particular, on $\mathcal{P}_n$ we must have

$$dh \in \text{span}\,\{dh_i\}$$

contrary to the assertion that $U \cap \mathcal{P}_n$ be open and dense in $\mathcal{P}_n$.

So far, we have recovered (cf. Corollary 5.3) a necessary condition for an initial condition (α, γ) to generate an asymptotically convergent trajectory under the dynamics of the fast filtering algorithm; viz. the pseudo-polynomial $D(z, z^{-1})$ determined by (α, γ) must be sign-definite. In the case $n = 1$, it has been demonstrated in [5] using somewhat specialized, detailed analysis that, apart from initial data which can escape in finite time, this condition is also sufficient for global convergence. In the case $n = 2$, this has also been shown [6], although for $n \geq 2$ it is possible to have asymptotic convergence to equilibria which have a lower dimensional stable manifold (see Theorem 5.2) as well as to the unique Lyapunov stable equilibrium, corresponding to a stable factor of the pseudo-polynomial $D(z, z^{-1})$, as occurs in classical filtering. The final result in this section enables to determine at what points (α, γ) the invariant n-fold X_D defined by (5.9) is a smooth manifold. As it turns out, the singular points correspond to certain systems having a lower dimensional realization, which also are initial data converging to unstable equilibria.

Theorem 5.11. *Consider the n-fold X_D defined by* (5.9) *with a corresponding pseudo-polynomial $D(z, z^{-1})$. A point (α, γ) is a singular point of X_D if, and only if, $a(z)$ and $b(z)$, defined in terms of (α, γ) by* (3.12)*, have a common pair of reciprocal roots.*

To prove this result, we derive a general formula for the dimension of the tangent space $T_{(\alpha,\gamma)}X_D$ from which our assertion will follow by the implicit function theorem.

Lemma 5.12. $\dim T_{(\alpha,\gamma)}X_D = n + \sigma$*, where σ is the number of common pairs of reciprocal roots of the polynomials $a(z)$ and $b(z)$ given by* (3.12)*.*

Proof of Theorem 5.11. According to Lemma 5.6 and Lemma 5.7, the tangent space, $T_{(\alpha,\gamma)}X_D$, to X_D at (α, γ) has the same dimension as the vector space W of all solutions (p, q) to the variational equation (5.17), i.e. W is the subspace

$$W = \{(p, q) \mid S(a)q + S(b)p = 0\} \tag{5.27}$$

of $V_{n-1} \times V_n$, where $a(z)$ and $b(z)$ are given by (3.12). Now consider the map $\text{proj}_1 : W \to V_{n-1}$ defined via

$$\text{proj}_1(p, q) = p. \tag{5.28}$$

In particular,

$$\dim W = \dim \ker(\text{proj}_1) + \dim \text{range}\,(\text{proj}_1) \tag{5.29}$$

where

$$\ker(\text{proj}_1) = \{(0, q) \mid S(a)q = 0\} \simeq \ker S(a) \tag{5.30}$$

and

$$\text{range}\,(\text{proj}_1) = \{p \in V_{n-1} | S(b)p \in \text{range } S(a)\} \tag{5.31}$$

Now recall from Lemma 5.5 that

$$\dim \ker S(a) = \rho_1, \quad \dim \ker S(b) = \rho_2 \tag{5.32}$$

where ρ_1 and ρ_2 are the number of pairs of reciprocal roots $(\lambda, 1/\lambda)$ of $a(z)$ and $b(z)$ respectively, counted with multiplicity. Matters being so, we can also characterize the range of $S(a)$ [and that of $S(b)$] in the vector space of symmetric pseudo-polynomials $\mathcal{D}_n$.

Explicitly, if $(\lambda_1, 1/\lambda_1), \dots, (\lambda_{\rho_1}, 1/\lambda_{\rho_1})$ are the ρ_1 pairs of reciprocal roots (counted with multiplicity) of $a(z)$, then the range of $S(a)$ is the codimension ρ_1 subspace of symmetric pseudo-polynomials which vanish at $\{\lambda_1, \dots, \lambda_{\rho_1}, 1/\lambda_1, \dots, 1/\lambda_{\rho_1}\}$. With these notations

$$\dim \ker(\text{proj}_1) = \rho_1 \tag{5.33}$$

fixing the first term of (5.29). To determine the second, observe that the map $S(b)$ sends any $p \in \text{range proj}_1$ into the subspace

$$U := \text{range } S(a) \cap \text{range } S(b) \tag{5.34}$$

which consists of all pseudo-polynomials in $\mathcal{D}_n$ with $\rho_1 + \rho_2 - \sigma$ pairs of reciprocal zeros fixed, where σ is the number of common such pairs of $a(z)$ and $b(z)$. The space U has codimension $\rho_1 + \rho_2 - \sigma$ in V_n, i.e.

$$\dim U = n + 1 - \rho_1 - \rho_2 + \sigma. \tag{5.35}$$

Therefore, since $\dim \ker S(b) = \rho_2$, the dimensions of the subspace

$$Z := \{p \in V_n | S(b)p \in \text{range } S(a)\} \tag{5.36}$$

is $n + 1 - \rho_1 + \sigma$, and consequently

$$\dim \text{range proj}_1 = n - \rho_1 + \sigma, \tag{5.37}$$

i.e. one less than $\dim Z$, provided that there is a $p \in Z$ which does not belong to V_{n-1}. But this is the case because

$$S(b)a = S(a)b \in \mathcal{D}_n \tag{5.38}$$

i.e. $a(z) \in Z$. Combining (5.29), (5.33) and (5.37) we then see that $\dim W = n + \sigma$. $\square$

6. Fast Filtering Algorithm, Riccati Equations and Lagrangian Grassmannians

Our goal is to prove a global convergence theorem for the fast filtering algorithm (4.4), or, equivalently, (2.16). As is to be expected, the convergence of (4.4) is intimately connected to the convergence of the matrix Riccati equation (2.2). It is no restriction to assume that $\alpha_n \neq 0$. In fact, as noted above, if $\alpha_n = 0$ we can reduce the dimension of the dynamical system (4.4), replacing n by $k < n$, so that $\alpha_k \neq 0$.

Lemma 6.1. *Let $\alpha_n \neq 0$. Then, the fast filtering algorithm* (4.4) *tends to a limit* $(\alpha_\infty, 0)$ *if and only if the Riccati equation* (2.2) *with initial condition $P_0 = 0$ converges to some equilibrium P_∞. Here P_∞ satisfies the algebraic Riccati equation*

$$\Lambda(P) := FPF' - P + (g - FPh)(1 - h'Ph)^{-1}(g - FPh)' = 0. \tag{6.1}$$

where the parameters (F, g) are those corresponding to the initial condition (α, γ) of (4.4)*, and*

$$\alpha_\infty = (1 - h'P_\infty h)^{-1}(g - FP_\infty h) + a. \tag{6.2}$$

Proof. The Riccati equation (2.2) can be written as

$$P_{t+1} - P_t = \Lambda(P_t). \tag{6.3}$$

As shown in [25] and pointed out in [5], the structure of the fast filtering algorithm is reflected in the fact that initial condition $P_0 = 0$ renders $\Lambda(P_0) = gg'$ nonnegative definite and rank one, a property which is preserved along the trajectory so that

$$\Lambda(P_t) = r_t g(t)g(t)' \tag{6.4}$$

where

$$r_t = 1 - h'P_t h \tag{6.5}$$

If the fast filtering algorithm (4.4) converges, then by Lemma 5.1, $(\alpha(t), \gamma(t)) \to (\alpha_\infty, 0)$ for some $\alpha_\infty \in \mathbb{R}^n$, and, r_t tends to a limit r_∞, as $t \to \infty$. Hence, according to Corollary 3.4, $a(t) \to \alpha_\infty$ and $b(t) \to \alpha_\infty$ and consequently $g(t) := \frac{1}{2}[b(t) - a(t)] \to 0$. In view of (6.3) and (6.4), this implies that P_t tends to a limit P_∞ as $t \to \infty$. Conversely, suppose that $P_t \to P_\infty$ as $t \to \infty$. Then $\Lambda(P_t) \to 0$, and, by (6.5), $r_t \to r_\infty$. The condition $\alpha_n = a_n + g_n \neq 0$ implies that $r_\infty \neq 0$. In fact, if $r_\infty = 0$, i.e. $h'P_\infty h = 1$, convergence would require that $g = FP_\infty h$ and consequently that $g_n = -a_n h'P_\infty h = -a_n$ (see (2.10)), contradicting the assumption that $\alpha_n \neq 0$. Therefore, by (6.4), $g(t)g(t)'$ and hence g(t) tends to zero, which, in turn, implies that $\gamma_t = g_1(t) \to 0$, i.e. $\gamma(t) \to 0$. Then it follows from (4.4a) that $\alpha(t)$ tends to a limit α_∞ as $t \to \infty$. Finally, we see from (2.17) and (4.1) that the Kalman gain tends to $k_\infty := \alpha_\infty - a$ as $t \to \infty$. On the other hand, it follows from (2.8) that $k_\infty = r_\infty^{-1}(g - FP_\infty h)$ and hence (6.2) holds. □

Lemma 6.2. *The statement of Lemma 6.1 remains true if $F := J - ah'$ defined by* (2.10)*, is replaced by $F := J - bh'$. If $\alpha_n \neq 0$, then at least one of the matrices $(J - ah')$ and $(J - bh')$ is nonsingular.*

Proof. Exchanging the roles of a and b amounts to changing the initial condition (α, γ) of the dynamical system (4.4) for $(\alpha, -\gamma)$. If (α, γ) has the orbit $\{(\alpha(t), \gamma(t))\}$, then a simple inspection of (4.4) shows that $(\alpha, -\gamma)$ has the orbit $\{(\alpha(t), -\gamma(t))\}$ so that (α, γ) converges if and only if $(\alpha, -\gamma)$ does, both tending to the same limit $(\alpha_\infty, 0)$. This proves the first part of the lemma. To prove the second part, suppose that both $(J - ah')$ and $(J - bh')$ are singular, i.e. $a_n = b_n = 0$. Then $\alpha_n = \frac{1}{2}(a_n + b_n) = 0$ contradicting the assumption that $\alpha_n \neq 0$. □

Lemma 6.3. *Suppose F is nonsingular. Then $h'F^{-1}g = 1$ if and only if $\alpha_n = 0$.*

Proof. Let $v(z)$ be as in (2.1). Then

$$v(0) = \frac{1}{2} - h'F^{-1}g$$

On the other hand, since $\gamma_t = -\varphi_{t+1}(0) = \psi_{t+1}(0)$ for $t = 0, 1, 2, \dots$ and $\varphi_0 = \psi_0 = 1$, (3.9) yields

$$v(0) = -\frac{1}{2} \frac{\gamma_{n-1} + \alpha_1 \gamma_{n-2} + \cdots + \alpha_{n-1}\gamma_0 - \alpha_n}{\gamma_{n-1} + \alpha_1 \gamma_{n-2} + \cdots + \alpha_{n-1}\gamma_0 + \alpha_n}$$

which equals $-\frac{1}{2}$ if and only if $\alpha_n = 0$. □

In considering the algebraic Riccati equation (6.1) it is important to remember that the situation here is more general than that usually considered in Kalman filtering (where $v(z)$ is positive real) since F may be unstable and $r_\infty := 1 - h'P_\infty h$ may be negative. Here the symmetric matrix P_∞ may have both negative and zero eigenvalues.

Recall now that there is an extensive literature, see e.g. [2, 29, 30, 35], on the solution of a matrix Riccati equation as a power iteration on the *Lagrangian Grassmannian* manifold, $LG(n, 2n)$ consisting of n-dimensional subspaces $\mathcal{U} \subset \mathbb{R}^{2n}$ which are *Lagrangian* in the sense that $x'\left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]y = 0$ for all $x, y \in \mathcal{U}$. In regard to (2.2) this amounts to noting first the well-known fact that the dynamics of the matrix Riccati equation can be described via a linear fractional transformation. Note that, in view of Lemma 6.2 and Lemma 6.3, it is no restriction to assume that F is nonsingular and that the parameter σ, defined in Proposition 6.4, is nonzero in analyzing the convergence of (4.4).

Proposition 6.4. *The matrix Riccati recursion* (2.2) *may be reformulated as*

$$P_{t+1} = (S_{21} + S_{22}P_t)(S_{11} + S_{12}P_t)^{-1} \tag{6.6}$$

where the $2n \times 2n$ matrix

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{6.7}$$

is the symplectic matrix

$$S = \begin{bmatrix} F^{-1} + F^{-1}gh'F^{-1}\sigma^{-1} & F^{-1}gg'\sigma^{-1} \\ -hh'F^{-1}\sigma^{-1} & F' - hg'\sigma^{-1} \end{bmatrix}' \tag{6.8a}$$

with

$$\sigma = 1 - h'F^{-1}g. \tag{6.8b}$$

Proof. A straightforward calculation shows that $\Lambda(P)$, defined by (6.1), may be written

$$\Lambda(P) := APA' - P + APh(1 - h'Ph)^{-1}h'PA' + gg' \tag{6.9}$$

where $A := F - gh'$. Since F is invertible, so is A. In fact,

$$A^{-1} = F^{-1} + F^{-1}gh'F^{-1}\sigma^{-1} \tag{6.10}$$

Consequently, (6.3) and the fact that

$$(I - hh'P)^{-1} = I + (1 - h'Ph)^{-1}hh'P \tag{6.11}$$

implies that

$$\begin{aligned} P_{t+1} &= gg' + AP_t(I - hh'P_t)^{-1}A' \\ &= [gg'(A')^{-1}(I - hh'P_t) + AP_t](I - hh'P_t)^{-1}A' \end{aligned}$$

which yields (6.6) with $S_{11} = (A')^{-1}$, $S_{12} = -(A')^{-1}hh'$, $S_{21} = gg'(A')^{-1}$ and $S_{22} = A + (1-\sigma^{-1})gh'$. Inserting (6.10) then yields (6.8). A simple calculation shows that S is symplectic, i.e. that

$$S'\hat{J}S = \hat{J} \tag{6.12a}$$

where

$$\hat{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \quad \square \tag{6.12b}$$

Corollary 6.5. *The algebraic Riccati equation* (6.1) *may be written in the alternative form*

$$P = APA' + APh(1-h'Ph)^{-1}h'PA' + gg' \tag{6.13}$$

where $A := F - gh'$ is invertible, and, in terms of A, the symplectic matrix S takes the form

$$S = \begin{bmatrix} (A')^{-1} & -(A')^{-1}hh' \\ gg'(A')^{-1} & A + (1-\sigma^{-1})gh' \end{bmatrix} \tag{6.14}$$

Next, setting

$$P_t = Y_t X_t^{-1} \tag{6.15}$$

and applying Proposition 6.4, we see that the matrix Riccati equation may be viewed as a linear symplectic system

$$Z_{t+1} = SZ_t \tag{6.16}$$

where $Z_t = \left[\begin{smallmatrix} X_t \\ Y_t \end{smallmatrix}\right]$ and $Z_0 = \left[\begin{smallmatrix} I \\ P_0 \end{smallmatrix}\right]$. In particular, Lemma 6.1 states that the dynamics of the fast filtering algorithm corresponds to the initial condition $P_0 = 0$, i.e.

$$Z_0 = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]. \tag{6.17}$$

Studying the linear system (6.16) on the manifold $LG(n, 2n)$ of Lagrangian subspaces in $\mathbb{R}^{2n}$ instead of (2.2) or (4.4) amounts to a compactification of the phase space in the sense that P_t is also allowed to take infinite values, corresponding to X_t being singular. In particular, this compactification provides a model in which we can analyze high-gain limits, as well as finite escape, of the sequence of Kalman gains. The fact that P_t is symmetric insures that the subspace spanned by the columns of $\left[\begin{smallmatrix} X_t \\ Y_t \end{smallmatrix}\right]$ is Lagrangian.

In view of this, the dynamical behavior of the Riccati equation (2.2) as well as the fast algorithm (2.16), or (4.4), depends on the eigenvalue structure of S, which is connected to the zero structure of the pseudo-polynomial $D(z, z^{-1})$ through the following proposition.

Proposition 6.6. *Let $\alpha_n \neq 0$. Then the eigenvalues of S are identical to the zeros of the pseudo-polynomial $D(z, z^{-1})$.*

Proof. Since $\alpha_n \neq 0$, we have $\kappa_n \neq 0$. By a straightforward computation, we see that the characteristic polynomial of S is

$$\begin{aligned} \chi_S(z) = z^{2n} + \frac{\kappa_{n-1}}{\kappa_n} z^{2n-1} + \frac{\kappa_{n-2}}{\kappa_n} z^{2n-2} + \cdots + \frac{\kappa_1}{\kappa_n} z^{n+1} + \frac{2}{\kappa_n} z^n \\ + \frac{\kappa_1}{\kappa_n} z^{n-1} + \cdots + \frac{\kappa_{n-2}}{\kappa_n} z^2 + \frac{\kappa_{n-1}}{\kappa_n} z + 1 \end{aligned}$$

where $\kappa_1, \kappa_2, \ldots, \kappa_n$ are integral constants defined in Theorem 5.4. Comparing this with (4.10) and (5.7) we see that

$$D(z, z^{-1}) = \alpha_n z^{-n} \chi_S(z). \tag{6.18}$$

from which the proposition follows. $\square$

This proposition allows us to analyze the dynamics of (2.2) and (4.4) not only for parameters (or, in the case of (4.4), initial conditions) that satisfy the positive real condition but for general choices of parameters (initial conditions). If $\alpha_n = 0$, Proposition 6.6 should be applied to the dimension-reduced problem mentioned above. Hence, in this case the Riccati equation (2.2) can be replaced by one of smaller dimension, which is actually due to the occurrence of invariant directions [3], as was pointed out in [27] and further developed in [31].

The basic question now is to determine under what conditions (6.16) converges, i.e. under what conditions $S^t\mathcal{Z}_0$ tends to a limit as $t \to \infty$, where $\mathcal{Z}_0$ is the subspace spanned by the columns of Z_0, i.e. $\mathcal{Z}_0 = \operatorname{Im}\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$. Let us first study the set of equilibria of the power iteration $S^t\mathcal{Z}_0$, which must clearly consist of those n-dimensional subspaces

$$\mathcal{U} = \operatorname{Im}\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]; \qquad X, Y \;\; n \times n \tag{6.19}$$

which are S-invariant. In order that $\mathcal{U}$ should correspond to a (finite) solution of algebraic Riccati equation (6.1), as required by Lemma 6.1, $\mathcal{U}$ must be such that X is nonsingular so that

$$P = YX^{-1} \tag{6.20}$$

can be formed, and, for P to be the limit of the sequence $\{P_t\}$, $\mathcal{U}$ must be Lagrangian so that P is symmetric. The following is consequence of (h, F) being observable.

Lemma 6.7. *Let $\alpha_n \neq 0$ and let $\mathcal{U}$, defined by* (6.19), *be Lagrangian. Then X is nonsingular.*

For the proof, we shall need a result which is a discrete-time version of a result due the Kučera [23]; see [34, p. 379]. Since it is surprisingly more complicated than the continuous-time result, and we shall need it again below, we state it as a lemma, the proof of which is deferred to the Appendix.

Lemma 6.8. *Let $\alpha_n \neq 0$ and let $\mathcal{U}$ be an n-dimensional S-invariant Lagrangian subspace. Then, the subspace $\mathcal{W} := \mathcal{U} \cap \operatorname{Im}\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ satisfies the invariance condition*

(i) $S\mathcal{W} \subset \mathcal{W}$

and, which is equivalent, the reversed invariance condition

(ii) $S^{-1}\mathcal{W} \subset \mathcal{W}$

The same statements holds for $\tilde{\mathcal{W}} := \mathcal{U} \cap \operatorname{Im}\left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$.

Now the proof of Lemma 6.7 follows along the lines of the proof of Shayman's Proposition 1 in [34].

Proof of Lemma 6.7. Suppose $\tilde{\mathcal{W}}$, defined in Lemma 6.8, has dimension k, and that X is singular so that $k > 0$. Then

$$\tilde{\mathcal{W}} = \operatorname{Im}\begin{bmatrix} 0 \\ V \end{bmatrix}$$

for some $n \times k$ matrix V. Since $S\tilde{\mathcal{W}} \subset \tilde{\mathcal{W}}$ (Lemma 6.8) and S is nonsingular (Proposition 6.6), there is a nonsingular $k \times k$ matrix T such that

$$S\begin{bmatrix} 0 \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ V \end{bmatrix} T$$

i.e., $-(A')^{-1}hh'V = 0$ and $AV + (1 - \sigma^{-1})gh'V = VT$. The first of these equations yields

$$h'V = 0 \tag{6.21a}$$

whereupon the second becomes

$$AV = VT \tag{6.21b}$$

However, since (h, F) is observable, so is (h, A), for $A = F - gh'$. Therefore (6.21) implies that $V = 0$, contradicting the assumption that X is singular. □

A partial answer to the question whether the power iteration $S^t\mathcal{Z}_0$ converges can now be given by the following lemma, which generalizes some results due to Parlett and Poole [30]. This requires a few definitions. For any linear operator $A : \mathbb{R}^m \to \mathbb{R}^m$, an A-invariant subspace $\mathcal{U}$ is *dominant* (*codominant*) if the eigenvalues of the restriction $A|\mathcal{U}$ have moduli greater than or equal to (smaller than or equal to) those of all other eigenvalues of A.

Lemma 6.9. *Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be a linear operator. If there is a unique p-dimensional dominant A-invariant subspace $\mathcal{U}^-$ and a unique $(m-p)$-dimensional codominant A-invariant subspace $\mathcal{U}^+$, then $A^t\mathcal{X} \to \mathcal{U}^-$ as $t \to \infty$ for each p-dimensional subspace $\mathcal{X}$ such that $\mathcal{X} \cap \mathcal{U}^+ = 0$.*

Proof. If the eigenvalues of A (counted with multiplicity) satisfy

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_p| > |\lambda_{p+1}| \geq \cdots \geq |\lambda_m|,$$

then the statement of the lemma follows directly from Theorem 4 in [30]. On the other hand, if

$$|\lambda_1| \geq \cdots \geq |\lambda_{p-q}| > |\lambda_{p-q+1}| = \cdots = |\lambda_{p+r}| > |\lambda_{p+r+1}| \geq \cdots \geq |\lambda_m|,$$

[or there is no eigenvalue larger (smaller) in modulus than λ_p, in which case we set $q = p$ $(r = m - p)$], we define $\hat{\mathcal{U}}^-$ and $\hat{\mathcal{U}}^+$ to be the subspaces spanned by the generalized eigenvectors corresponding to $\{\lambda_1, \ldots, \lambda_{p-q}\}$ and $\{\lambda_{p-q+1}, \ldots, \lambda_m\}$ respectively. Moreover, let $\tilde{\mathcal{U}}^-$ and $\tilde{\mathcal{U}}^+$ be the subspaces spanned by the generalized eigenvectors in $\mathcal{U}^-$ respectively $\mathcal{U}^+$ corresponding to eigenvalues of modulus $|\lambda_p|$. Then $\tilde{\mathcal{U}}^-$ and $\tilde{\mathcal{U}}^+$ are A-invariant subspaces of $\mathcal{U}^-$ and $\mathcal{U}^+$ of dimensions q and r respectively. In fact, $\dim(\tilde{\mathcal{U}}^- \cap \tilde{\mathcal{U}}^+) = \min(q, r)$. Now, since $\mathcal{X} \cap \mathcal{U}^+ = 0$, $\dim \mathcal{X} = p$ and $\dim \mathcal{U}^+ = m - p$, $\mathbb{R}^m = \mathcal{X} \oplus \mathcal{U}^+$, where $\oplus$ denotes direct sum. Therefore, since $\mathcal{U}^+ \subset \hat{\mathcal{U}}^+$, there is a subspace $\tilde{\mathcal{X}} \subset \mathcal{X}$ of dimension q such that $\hat{\mathcal{U}}^+ = \tilde{\mathcal{X}} \oplus \mathcal{U}^+$. Let $\hat{\mathcal{X}}$ be any $(p - q)$-dimensional subspace of $\mathcal{X}$ such that $\mathbb{R}^m = \hat{\mathcal{X}} \oplus \hat{\mathcal{U}}^+$. Now, since $\hat{\mathcal{U}}^-$ is the unique dominant $(p - q)$-dimensional A-invariant subspace, and $\hat{\mathcal{U}}^+$ is an invariant complement which, by construction satisfies $\hat{\mathcal{X}} \cap \hat{\mathcal{U}}^+ = 0$, $A^t\hat{\mathcal{X}} \to \hat{\mathcal{U}}^-$ as $t \to \infty$ by Theorem 4 in [30]. Moreover, $\mathcal{X} \cap \mathcal{U}^+ = 0$ implies that $\tilde{\mathcal{X}} \cap \tilde{\mathcal{U}}^+ = 0$. Then following the argument in the proof of Theorem 7 of [30], we see that $A^t\tilde{\mathcal{X}}$ becomes disjoint from the subspace corresponding to the eigenvalues $\{\lambda_{p+r+1}, \ldots, \lambda_m\}$ as $t \to \infty$ as these are smaller in modulus than $|\lambda_p|$. Therefore, since $\tilde{\mathcal{X}} \subset \hat{\mathcal{U}}^+$, the question of convergence of $A^t\tilde{\mathcal{X}}$ is reduced to that of Theorem 6 in [30] dealing with the equimodular case. Hence, because $\tilde{\mathcal{U}}^-$ $[\tilde{\mathcal{U}}^+]$ is the unique q-dimensional [r-dimensional] dominant A-invariant subspace of $\hat{\mathcal{U}}^+$ and $\tilde{\mathcal{X}} \cap \tilde{\mathcal{U}}^+ = 0$, $A^t\tilde{\mathcal{X}} \to \tilde{\mathcal{U}}^-$ as $t \to \infty$. Consequently, since $\hat{\mathcal{X}} \oplus \tilde{\mathcal{X}} = \mathcal{X}$ and $\hat{\mathcal{U}}^- \oplus \tilde{\mathcal{U}}^- = \mathcal{U}^-$, $A^t\mathcal{X} \to \mathcal{U}^-$ as $t \to \infty$, as claimed. □

The following lemma shows that the basic assumptions of Lemma 6.9 are fulfilled for the power iteration $S^t\mathcal{Z}_0$, provided $D(z, z^{-1})$ is sign definite, i.e. has no zeros of odd multiplicity on the unit circle (Proposition 6.6). First, let us introduce some notations. Following Parlett and Poole [30] let us order the $2n$ generalized eigenvectors of S first by modulus of the associated eigenvalue with the largest first. Generalized eigenvectors whose eigenvalues have the same modulus are ordered by exponent, where the *exponent* $e(v)$ of a generalized eigenvector v is defined as

$$e(v) = m - 2g + 1 \tag{6.22}$$

where m is the *multiplicity* of v, i.e. the dimension of the smallest invariant subspace containing it, and g is the *grade* of v, i.e. the dimension of the largest cyclic subspace containing v. Thus let

$$v_1, v_2, \dots, v_{2n} \tag{6.23}$$

be the generalized eigenvectors ordered in this way, and let

$$\lambda_1, \lambda_2 \dots, \lambda_{2n} \tag{6.24}$$

the corresponding eigenvalues (which may be repeated). Then, for each $k = 1, 2, \dots, 2n$, $\operatorname{span}\{v_1, v_2, \dots, v_k\}$ is a dominant S-invariant subspace.

Lemma 6.10. *If S has no eigenvalues of odd multiplicity on the unit circle, there is a unique dominant n-dimensional S-invariant subspace $\mathcal{U}_D^-$ and a unique codominant n-dimensional subspace $\mathcal{U}_D^+$. Both are Lagrangian. In particular, $\mathcal{U}_D^-$ is spanned by $\{v_1, v_2, \dots, v_n\}$ in* (6.23).

Proof. With the generalized eigenvectors of S and its corresponding eigenvalues ordered as in (6.23) - (6.24), $\mathcal{U}_{\mathcal{D}}^- := \operatorname{span}\{, , \dots, \backslash\}$ is the *unique* dominant S-invariant n-subspace if either

(i) $|\lambda_n| > |\lambda_{n+1}|$

or

(ii) $|\lambda_n| = |\lambda_{n+1}|$ but $e(v_n) > e(v_{n+1})$

see [30, p. 404]. Now, recall that S is symplectic so that if λ is an eigenvalue then so is $1/\lambda$. Therefore, if S has no eigenvalues on the unit circle, then case (i) holds, so there is a unique dominant S-invariant n-subspace. If there are eigenvalues on the unit circle, there must be an even number, say $2q$ where $q \leq n$, so that $\{v_1, v_2, \dots, v_n\}$ contains $n - q$ generalized eigenvectors whose eigenvalues have moduli greater than 1 and q whose eigenvalues lie on the unit circle. If we can show that $e(v_n) > e(v_{n+1})$, case (ii) holds and there is a unique dominant S-invariant n-space, namely $\{v_1, v_2, \dots, v_n\}$. To this end, let $\mu_1, \mu_2, \dots, \mu_k$ be the eigenvalues of S on the unit circle (now *not* repeated), and let $m_1, m_2, \dots, m_k$ be their multiplicities. Then, $\sum_{i=1}^k m_i = 2q$. For each $i = 1, \dots, k$ let $v_i^{(j)}$, $j = 1, 2, \dots, m_i$, be the (generalized) eigenvectors corresponding to μ_i. The exponent of $v_i^{(j)}$ is

$$e(v_i^{(j)}) = m_i - 2j + 1.$$

Since, by assumption there are no eigenvalues of odd multiplicity on the unit circle, i.e. m_i is even, $e(v_i^{(j)}) \neq 0$. Therefore, $e(v_i^{(j)})$ is positive for $j = 1, 2, \dots, m_i/2$ and negative for $j = m_i/2 + 1, \dots, m_i$, and hence $\mathcal{U}_D^-$ is unique. In the same way, it is seen that $\mathcal{U}_D^+ := \operatorname{span}\{v_{n-q+1}, \dots, v_n, v_{n+q+1}, \dots, v_{2n}\}$ is the unique codominant S-invariant n-subspace. The proof that $\mathcal{U}_D^-$ and $\mathcal{U}_D^+$ are Lagrangian can be found in the appendix of [7] (Lemma 3.1x). $\square$

Returning to the fast filtering algorithm (4.4), the following lemma establishes the proper interpretation of the convergence of $S^t\mathcal{Z}_0$ to the dominant S-invariant subspace.

Lemma 6.11. *Let $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be the eigenvalues (counted with multiplicity) corresponding to the dominant S-invariant subspace $\mathcal{U}_D^-$ of Lemma 6.10, and let $\mathcal{Z}_0$ be the n-dimensional subspace spanned by the columns of* (6.17). *Then, if $S^t\mathcal{Z}_0 \to \mathcal{U}_D^-$, either the* *trajectory of* (4.4) *escapes to infinity in finite time or* $(\alpha(t), \gamma(t)) \to (\alpha_\infty, 0)$ *where the zeros of the corresponding polynomial*

$$\alpha_\infty(z) = z^n + \alpha_{\infty 1} z^{n-1} + \cdots + \alpha_{\infty n}$$

all lie in the closed unit disc. More precisely,

$$\alpha_\infty(z) = (z - \frac{1}{\lambda_1})(z - \frac{1}{\lambda_2}) \cdots (z - \frac{1}{\lambda_n}). \tag{6.25}$$

Proof. To say that $S^t \mathcal{Z}_0 \to \mathcal{U}_D^-$ is equivalent to say that

$$\left[\begin{smallmatrix} X_t \\ Y_t \end{smallmatrix}\right] \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] = (v_1, v_2, \ldots, v_n) T \tag{6.26}$$

for some nonsingular $n \times n$ matrix T, where, as above, $\left[\begin{smallmatrix} X_t \\ Y_t \end{smallmatrix}\right] = S^t \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$. Since $\mathcal{U}_D^-$ is Lagrangian, X is nonsingular (Lemma 6.7). Therefore, if X_t is nonsingular for all $t \in \mathbb{Z}$, the solution $P_t = Y_t X_t^{-1}$ of the matrix Riccati equation (2.2) with initial condition $P_0 = 0$ tends to the limit $P = Y X^{-1}$, which is thus a real symmetric solution of the algebraic Riccati equation (6.1). Then, by Lemma 6.1, $(\alpha(t), \gamma(t)) \to (\alpha_\infty, 0)$ where

$$\alpha_\infty = (1 - h' P h)^{-1} (a + g - JPh) \tag{6.27}$$

If, on the other hand, X_t becomes singular in finite time τ, the Riccati trajectory $P_t = Y_t X_t^{-1}$ escapes to infinity at time τ. To analyze the convergent case, first note that T cancels out in forming $P_t = Y_t X_t^{-1}$ and $P = Y X^{-1}$ and therefore we may without restriction assume that $T = I$. Hence

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \Lambda \tag{6.28}$$

where Λ is the block diagonal matrix formed by the Jordan blocks corresponding to $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. From this it follows that

$$S_{11} + S_{12} P = X \Lambda X^{-1} \tag{6.29}$$

Now, substituting S_{11} and S_{12} in (6.29) for their values as defined in (6.8), we have

$$(S_{11}' + P S_{12}')^{-1} = F[F + \sigma^{-1}(g - FPh)h']^{-1} F, \tag{6.30}$$

to which we apply the well-known "matrix inversion lemma"

$$(A + BD^{-1}C)^{-1} = A^{-1} - A^{-1}B(D + CA^{-1}B)^{-1}CA^{-1} \tag{6.31}$$

to obtain

$$(S_{11}' + P S_{12}')^{-1} = F - (1 - h'Ph)^{-1}(g - FPh)h' = J - \alpha_\infty h' \tag{6.32}$$

Therefore, setting $\Gamma := J - \alpha_\infty h'$, (6.29) and (6.32) yield

$$(\Gamma')^{-1} X = X \Lambda \tag{6.33}$$

i.e. $(\Gamma')^{-1}$ has eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then, $\alpha_\infty(z)$ being the characteristic polynomial of Γ must have the form (6.25), as claimed. Since $|\lambda_i| \geq 1$ for $i = 1, 2, \ldots, n$, the zeros of $\alpha_\infty(z)$ are all in the closed unit disc. □

Finally, to establish a global convergence theorem for the fast filtering algorithm (4.4) based on Lemma 6.9, it therefore remains to interpret the condition $\mathcal{U}_D^+ \cap \mathcal{Z}_0 = 0$, where $\mathcal{Z}_0$ is the initial space corresponding to $Z_0 = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, in terms of the parameters (i.e. the initial conditions) of the algorithm.

Lemma 6.12. *Let* $\alpha_n \neq 0$ *and let* $\mathcal{U}$ *be an* n*-dimensional Lagrangian* S*-invariant subspace, and set* $\mathcal{Z}_0 := \operatorname{Im} \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$. *Then, if* $a(z)$ *and* $b(z)$ *are coprime,* $\mathcal{U} \cap \mathcal{Z}_0 = 0$.

Proof. As above, set $\mathcal{W} := \mathcal{U} \cap \mathcal{Z}_0$ and let U be a full-rank matrix such that $\mathcal{W} = \text{Im}\left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right]$. Then, since $\mathcal{W}$ is S-invariant (Lemma 6.8), there is a square matrix T such that $S\left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right]T$. Therefore, in view of (6.14), $(A')^{-1}U = UT$ and $gg'(A')^{-1}U = 0$, from which we see that

$$U'A^{-n}[g, Ag, \dots, A^{n-1}g] = 0 \tag{6.34}$$

Consequently $U = 0$, i.e. $\mathcal{W} = 0$, if and only if (A, g) is reachable. Since $A = F - gh'$, this is equivalent to (F, g) being reachable. However, since

$$\frac{1}{2}\frac{b(z)}{a(z)} = h'(zI - F)^{-1}g + \frac{1}{2}$$

and (h, F) is observable, (F, g) is reachable if and only if $a(z)$ and $b(z)$ are coprime. □

7. Global Convergence of the Fast Filtering Algorithm

We are now in a position to formulate the global convergence theorem. To this end, let $\mathcal{D}$ be the subset of all $(\alpha, \gamma) \in \mathbb{R}^{2n}$ such that $D(z, z^{-1})$ is sign definite on the unit circle, i.e. either nonnegative or nonpositive there. Finally, denote by Ω_e the subset of initial conditions $(\alpha, \gamma) \in \mathbb{R}^{2n}$ which generate trajectories which escape in finite time.

Theorem 7.1. *For initial conditions $(\alpha, \gamma) \in \mathbb{R}^{2n} - \Omega_e$ there is convergence to an equilibrium under the dynamics of the fast filtering algorithm if and only if the corresponding pseudo-polynomial $D(z, z^{-1})$ is sign definite. More precisely, the following statements hold:*

(i) *Ω_e and $\overline{\mathcal{D} \cap \Omega_e}$ have Lebesgue measure zero.*

(ii) *$(\alpha, \gamma) \in \mathcal{D} - \mathcal{D} \cap \Omega_e$ is a necessary and sufficient condition for convergence to an equilibrium.*

(iii) *If $(\alpha, \gamma) \in \mathcal{D} - \mathcal{D} \cap \Omega_e$, then $(\alpha_t, \gamma_t) \to (\alpha_\infty, 0)$ and the corresponding limit polynomial*

$$\alpha_\infty(z) = z^n + \alpha_{\infty 1}z^{n-1} + \dots + \alpha_{\infty n}$$

satisfies

$$\alpha_\infty(z) = \tilde{\alpha}_\infty(z)\theta(z)$$

where $\tilde{\alpha}_\infty(z)$ has all its zeros in the closed unit disc and where

$$\theta(z) = (a, b),$$

i.e. $\theta(z)$ is the greatest common divisor of $a(z)$ and $b(z)$.

Moreover, $\tilde{\alpha}_\infty(z)$ is determined up to a nonzero, scalar multiplicative factor r_∞ by the spectral factorization problem

$$\tilde{a}(z)\tilde{b}(1/z) + \tilde{a}(1/z)\tilde{b}(z) = r_\infty\tilde{\alpha}_\infty(z)\tilde{\alpha}_\infty(1/z). \tag{7.1a}$$

where

$$a(z) = \tilde{a}(z)\theta(z), \quad b(z) = \tilde{b}(z)\theta(z). \tag{7.1b}$$

Theorem 7.1 not only characterizes those initial conditions which generate a convergent trajectory, but also provides for an explicit determination of the equilibrium to which the corresponding trajectory will converge. Conversely, from this explicit recipe we can also determine which initial conditions will generate a trajectory which converges to a given equilibrium.

Corollary 7.2. *In the notation of Theorem 7.1, suppose $\alpha_\infty(z) = \tilde{\alpha}_\infty(z)\theta(z)$ where $\tilde{\alpha}_\infty(z)$ is a Schur polynomial and $\theta(z)$ has all of its zeros in $|z| > 1$. Then, the global stable "manifold" $W^s(\alpha_\infty, 0)$ is given by*

$$W^s(\alpha_\infty, 0) = \{(\alpha, \gamma) \notin \Omega_e : \text{ (7.1) hold with } (a, b) \text{ given by (4.1) and } (\tilde{a}, \tilde{b}) = 1\}.$$

Similarly, the global unstable "manifold" can be parameterized as all coprime pairs $(\bar{a}, \bar{b})$ *satisfying*

$$a(z) = \bar{a}(z)\tilde{\alpha}_\infty(z), \quad b(z) = \bar{b}(z)\tilde{\alpha}_\infty(z),$$

and

$$\bar{a}(z)\bar{b}(1/z) + \bar{a}(1/z)\bar{b}(z) = \bar{r}_\infty \theta(z)\theta(1/z).$$

Finally, a global center manifold $W^c(\alpha_\infty, 0)$ *is given by the equilibrium set* E.

Remark 7.3. The existence of stable and unstable manifolds as locally invariant immersed manifolds is of course a local result. In harmony with this, Lemma 5.8 gives a result characterizing $W^s(\alpha_\infty, 0)$ as a submanifold near $(\alpha_\infty, 0)$. In contrast, the description of $W^s(\alpha_\infty, 0)$ in the large as given in Corollary 7.2 does allow for singular points. These singular points are characterized in Theorem 5.11.

Proof of Theorem 7.1. We first assume that $\alpha_n \neq 0$ so that the pseudo-polynomial $D(z, z^{-1})$ has degree n and the symplectic matrix S is well-defined and nonsingular.

Finite escape occurs for precisely the initial conditions

$$Z_0 = S^{-t}\tilde{Z} \qquad t = 0, 1, 2, \ldots \tag{7.2}$$

for which $\tilde{X}$ is singular. In (α, γ)-space, $\tilde{X}$ being singular corresponds to $(\tilde{\alpha}, \tilde{\gamma})$ belonging to the two hyperplanes $\gamma_{n-1} = \pm 1$. Forming the union of the countably many iterates via (7.2) of these hyperplanes forms a set Ω_e which has measure zero. Here $\mathcal{D} \cap \Omega_e$ also has measure zero. We shall defer the proof that $\overline{\mathcal{D} \cap \Omega_e}$ is a set of measure zero.

Concerning the second part of the theorem, we already have established that it is necessary that $D(z, z^{-1})$ is sign definite for (4.4) to converge (Corollary 5.3). We prove the converse statement by proving assertion (iii). Suppose that $D(z, z^{-1})$ is sign definite, i.e. it has no zeros of odd multiplicity on the unit circle. Then Proposition 6.6 implies that S has no eigenvalues of odd multiplicity on the unit circle. But, this implies that there are unique dominant and codominant S-invariant n-dimensional subspaces $\mathcal{U}_D^-$ and $\mathcal{U}_D^+$ respectively, which are Lagrangian (Lemma 6.10), so that we can apply Lemma 6.9.

First, suppose that $a(z)$ and $b(z)$ are relatively prime. Then, if $\mathcal{Z}_0$ is the subspace spanned by the columns of $Z_0 = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, we have $\mathcal{U}_D^+ \cap \mathcal{Z}_0 = 0$ (Lemma 6.12) so that $S^t\mathcal{Z}_0 \to \mathcal{U}_D^-$ as $t \to \infty$ (Lemma 6.9). But this implies that $(\alpha(t), \gamma(t)) \to (\alpha_\infty, 0)$ (Lemma 6.11) where $\alpha_\infty(z)$ has all its zeros in the closed unit disc, unless there is finite escape.

Next, suppose that $a(z) = \tilde{a}(z)\theta(z)$ and $b(z) = \tilde{b}(z)\theta(z)$, where $\theta(z)$ is a nontrivial monic polynomial and $\tilde{a}(z)$ and $\tilde{b}(z)$ are relatively prime. Then the factor $\theta(z)$ can be canceled in $v(z) = \frac{1}{2}\frac{a(z)}{b(z)}$ so we may consider the dimension-reduced problem with (a, b) exchanged for $(\tilde{a}, \tilde{b})$. Since $D(z, z^{-1}) = \tilde{D}(z, z^{-1})|\theta(z)|^2$ is sign definite on the unit circle, then so is $\tilde{D}(z, z^{-1})$, and consequently, unless there is finite escape,

$$(\tilde{\alpha}(t), \tilde{\gamma}(t)) \to (\tilde{\alpha}_\infty, 0) \tag{7.3}$$

as $t \to \infty$, where $\tilde{\alpha}_\infty(z)$ has all its zeros in the closed unit disc (Lemma 6.11). On the other hand, it was shown in [27], and further elaborated upon in [5], that the fast algorithm (2.16), which is equivalent to (4.4), can be written in the Szegö-like polynomial form

$$\begin{cases} Q_{t+1}(z) = Q_t(z) - \gamma_t z Q_t^*(z) \\ Q_{t+1}^*(z) = zQ_t^*(z) - \gamma_t Q_t(z) \end{cases} \tag{7.4}$$

(see Section 2 in [5]), which through the transformation

$$\begin{cases} a_t(z) = r_t^{-1}[Q_t(z) - Q_t^*(z)] \\ b_t(z) = r_t^{-1}[Q_t(z) + Q_t^*(z)] \end{cases} \tag{7.5}$$

and (3.7) provides us with a polynomial version in (a, b)-coordinates of the dynamical system (4.4). From this we see that, if $a_0(z) := a(z)$ and $b_0(z) := b(z)$ have a nontrivial common factor $\theta(z)$, then

$$\begin{cases} a_t(z) = \tilde{a}_t(z)\theta(z) \\ b_t(z) = \tilde{b}_t(z)\theta(z) \end{cases} \tag{7.6}$$

for *all* $t = 0, 1, 2, 3, \dots$. Since $\tilde{a}_0(z) = \tilde{a}(z)$ and $\tilde{b}_0(z) = \tilde{b}(z)$, it follows readily from (7.4) and (7.5), that $\{(\tilde{a}_t, \tilde{b}_t)\}$ is a trajectory in (a, b)-coordinates of the reduced system obtained by cancellation of the factor $\theta(z)$. In view of (7.3) and (A-1), $\tilde{a}_t(z) \to \tilde{\alpha}_\infty(z)$ and $\tilde{b}_t(z) \to \tilde{\alpha}_\infty(z)$ as $t \to \infty$, where $\alpha_\infty(z)$ has all its zeros in the closed unit disc. Therefore, $a_t(z) \to \tilde{\alpha}_\infty(z)\theta(z)$ and $b_t(z) \to \tilde{\alpha}_\infty(z)\theta(z)$ so that $(\alpha_t, \gamma_t) \to (\alpha_\infty, 0)$ where

$$\alpha_\infty(z) = \tilde{\alpha}_\infty(z)\theta(z), \tag{7.7}$$

the zeros of which are located in the closed unit disc if and only if $\theta(z)$, the common factor of $a(z)$ and $b(z)$, has all its zeros in the closed unit disc.

To complete the proof of (i), we shall now demonstrate that

$$\overline{\mathcal{D} \cap \Omega_e} \subset \mathcal{D} \cap \Omega_e \cup \mathcal{F}_1 \cup \mathcal{F}_2$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are sets of measure zero. Indeed, $\mathcal{F}_1$ is the algebraic set consisting of those pairs (α, γ) for which the corresponding polynomials (a, b) have a nontrivial common factor and $\mathcal{F}_2$ is the algebraic set consisting of those pairs for which the corresponding pseudo polynomial $D(z, z^{-1})$ has a double root. Suppose then that $(\alpha^{(n)}, \gamma^{(n)})$ is a sequence in $\mathcal{D} \cap \Omega_e$ with limit

$$\lim_{n\to\infty} (\alpha^{(n)}, \gamma^{(n)}) = (\alpha, \gamma).$$

Of course, $(\alpha, \gamma) \in \mathcal{D}$ so our claim will follow if we show that $(\alpha, \gamma) \in \mathcal{D} - \mathcal{D} \cap \Omega_e$ implies $(\alpha, \gamma) \in \mathcal{F}_1 \cup \mathcal{F}_2$. From (iii) of Theorem 7.1, we know that, if $(\alpha_0, \gamma_0) = (\alpha, \gamma)$ then

$$\lim_{t\to\infty} (\alpha_t, \gamma_t) = (\alpha_\infty, 0)$$

where $\alpha_\infty(z) = \tilde{\alpha}_\infty(z)\theta(z)$, where $\tilde{\alpha}_\infty(z)$ has all of its roots in the closed unit disc and $\theta = (a, b)$. If $\deg \theta \geq 1$ then $(\alpha, \gamma) \in \mathcal{F}_1$, so we suppose $\theta(z) \equiv 1$. In this case, to say $\alpha_\infty(z)$ has roots on the unit circle is to say $(\alpha, \gamma) \in \mathcal{F}_2$ so we may assume $\alpha_\infty(z)$ is a Schur polynomial, an assumption which we shall show is contrary to fact. If $\alpha_\infty(z)$ is a Schur polynomial, then $(\alpha_\infty, 0)$ belongs to the region $\mathcal{P}_n$ of all $(\alpha, \gamma) \in \mathbb{R}^{2n}$ satisfying the positive real conditions (2.13)-(2.15), and so $(\alpha_T, \gamma_T) \in \mathcal{P}_n$ for some finite $T > 0$. Since $\mathcal{P}_n$ is open and since the map on $\mathbb{R}^{2n}$

$$\Phi_T : (\alpha_0, \gamma_0) \mapsto (\alpha_T, \gamma_T),$$

defined by iterating the dynamical system (4.4) T times, is rational with no pole at (α, γ), there exists an $\varepsilon > 0$ such that

$$\Phi_T(B_\varepsilon(\alpha, \gamma)) \subset \mathcal{P}_n.$$

But then, no $(\alpha', \gamma') \in B_\varepsilon(\alpha, \gamma)$ can escape in finite time, contrary to the definition of (α, γ).

Finally, it is easy to modify the above argument to include the case $\alpha_n = 0$. Indeed, if for some $k < n$, $\alpha_n = \cdots = \alpha_{k+1} = 0$ and $\alpha_k \neq 0$, then $D(z, z^{-1})$ has degree k and the dynamical system (4.4) is reduced to a system of order $2k$ in $n - k$ steps. Therefore, $(\alpha(t), \gamma(t)) \to (\alpha_\infty, 0)$ if and only if $(\hat{\alpha}(t), \hat{\gamma}(t)) \to (\hat{\alpha}_\infty, 0)$, where the "hatted" quantities correspond to the reduced system. Then $\alpha_\infty(z) = z^{n-k}\hat{\alpha}_\infty(z)$ so that all statements concerning the reduced system also hold for the unreduced one. $\square$

We remark that the initial conditions (α, γ) for which the pseudo polynomial $D(z, z^{-1})$ fails to be sign definite, and there consequently is no convergence, form an unbounded open set in $\mathbb{R}^{2n}$. As we illustrate in the next section such points can be periodic or dense on some unbounded submanifold, depending on certain number theoretic considerations and leading to a remarkable sensitivity of the fast filtering algorithm to initial conditions in this region. We shall return to this topic in a subsequent paper.

8. **Examples and Simulations**

The purpose of this section is to illustrate our results for low order problems, particularly the cases $n = 1$ and $n = 2$. Since these cases have been treated in [5] and [6], respectively, we shall quote only those results which best illustrate our main theorem.

In the first-order case the dynamical system (4.4) takes the form

$$\alpha_{t+1} = \frac{\alpha_t}{1 - \gamma_t^2} \tag{8.1a}$$

$$\gamma_{t+1} = -\frac{\gamma_t \alpha_t}{1 - \gamma_t^2} \tag{8.1b}$$

corresponding to the rational function

$$v(z) = \frac{1}{2} \frac{b(z)}{a(z)} \tag{8.2}$$

where

$$\begin{aligned} a(z) &= z + \alpha - \gamma \\ b(z) &= z + \alpha + \gamma \end{aligned} \tag{8.3}$$

This case was studied in detail in [5], where it was shown that points (α, γ) in the interior of the diamond I, with corners $(\pm 1, 0)$, $(0, \pm 1)$, depicted in Figure 8.1 correspond to positive real $v(z)$, whereas the points (α, γ) in the shaded regions are precisely those for which $D(z, z^{-1})$ is sign definite on the unit circle, $v(1/z)$ being positive real in regions III and negative real in regions II. The dotted lines are the lines $\gamma = \pm 1$ of finite escape.

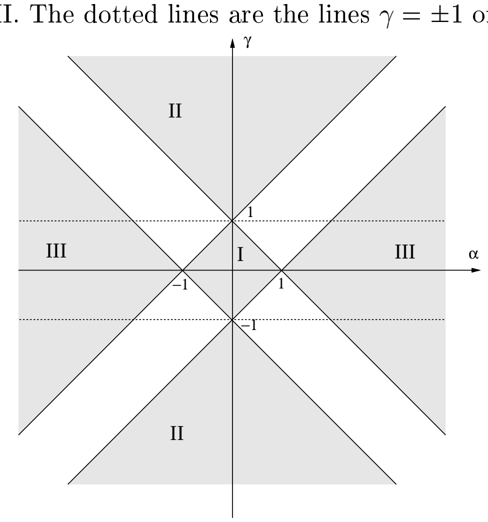

Figure 8.1

The invariant manifold X_D, defined by (5.9), becomes

$$1 + \alpha_t^2 - \gamma_t^2 = \frac{2}{\kappa}\alpha_t \tag{8.4}$$

valid for all $\kappa \neq 0$ [including $\kappa = \infty$, corresponding to $d_0(\alpha, \gamma) = 0$]; for $\kappa = 0$, the dynamical system (8.1) evolves along the axis $\alpha = 0$, converging in one step to the origin.

Figure 8.2 depicts the invariant manifolds defined by (8.4) for certain values of κ. For $\kappa^2 < 1$ these manifolds are hyperbolas completely contained in the shaded, sign-definite region, and for $\kappa^2 = 1$ they degenerate into a pair of intersecting lines, in the boundary of the shaded region, intersecting in $(1, 0)$ or $(-1, 0)$. In fact, each point in the shaded region lies on such an invariant manifold and converges to the intersection of this hyperbola with the segment $\{(\alpha, 0) \mid -1 \leq \alpha \leq 1\}$.

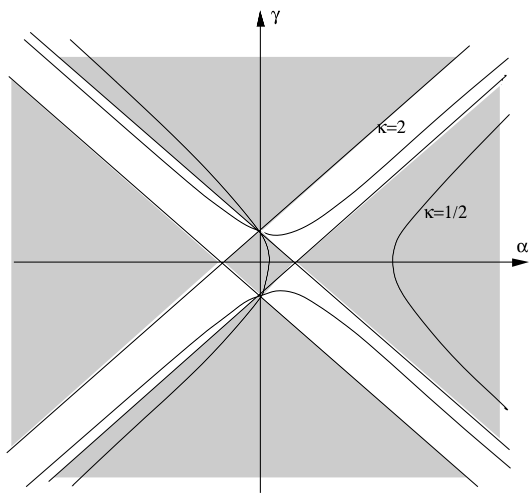

Figure 8.2

Since $a(z)$ and $b(z)$, displayed in (8.3), can have a common pair of reciprocal roots only in $(1, 0)$ and $(-1, 0)$, Theorem 5.27 states that these are the only singular points, σ in Lemma 5.12 being zero everywhere else, a fact that is illustrated by the above analysis.

It is easy to check (see [5] for details) that hyperbolas for which $\kappa^2 < 1$ correspond to those symplectic matrices S which have two real eigenvalues, one inside the unit circle and the other outside, whereas the case $\kappa^2 = 1$ yields a symplectic matrix S which has an eigenvalue of multiplicity 2 either at 1 or at -1. Convergence in these cases is therefore in accordance with Theorem 7.1, since S has no eigenvalue of odd multiplicity on the unit circle. On the other hand, if $\kappa^2 > 1$, there is a complex pair of such eigenvalues, and the hyperbolas lie in the white region of Figure 8.2 where the corresponding pseudo-polynomials $D(z, z^{-1})$ is sign indefinite on the unit circle, implying nonconvergence by Theorem 5.34.

Those hyperbolas for which $\kappa^2 < 1$ intersect the α-axis in two points, one of which is a point α_∞ so that the polynomial $z + \alpha_\infty$ is Schur; i.e. so that $|\alpha_\infty| < 1$. In [5] it has been shown that, not only is the hyperbola (8.4) locally a stable manifold, for $\kappa^2 < 1$, but rather it consists of a global stable manifold, excluding the unstable equilibrium and the measure zero set of points which escape in finite time. Also, $E = \{(\alpha_\infty, 0) : \alpha_\infty \in \mathbb{R}\}$ is a global center manifold through $(\alpha_\infty, 0)$. We note that the unstable equilibrium $(\alpha_\infty, 0)$, with $|\alpha_\infty| > 1$, has, by (5.2), a one-dimensional center manifold. In fact, these manifolds exist globally with the hyperbola being a global unstable manifold for the unstable equilibrium, on which trajectories either escape or evolve to the equilibrium, with the exception of the unstable equilibrium itself. E is again a global center manifold. The global convergence is completely understood in this case and described in [5].

If $\kappa^2 > 1$ then the hyperbolas do not intersect the α-axis, see Figure 8.2, and indeed the dynamics is far more complex. In fact, in [5] it is shown that there are two alternatives which taken together prove that (8.1) is sensitive to initial conditions, in the technical sense (as in [11]). Explicitly, one knows that either (A) or (B) holds:

(A) $\arctan\sqrt{\kappa^2-1} \in \mathbb{Q}\pi$ and hence

$$\frac{1}{2}\arctan\sqrt{\kappa^2-1} = \frac{q}{p}\pi \qquad \text{if } \kappa < -1$$

or

$$\frac{1}{2}\{\pi - \arctan\sqrt{\kappa^2-1}\} = \frac{q}{p}\pi \qquad \text{if } \kappa > 1,$$

where p and q are coprime natural numbers. If p is odd, $2(p-1)$ points on the hyperbola escape in finite time and if p is even there are $(p-2)$ such points. All other points are periodic with period p and every period p, $p \geq 3$, is possible.

(B) $\arctan\sqrt{\kappa^2-1} \notin \mathbb{Q}\pi$ and a countably infinite set of points on the hyperbola escape in finite time. All other points generate a dense orbit.

Finally, consider the points $(\pm 1, 0)$, correspondingly to $\kappa^2 = 1$. According to Theorem 5.2, the center manifold is two-dimensional and consequently is global. In fact, hyperbolas of all types, containing periodic orbits and dense orbits or consisting of stable and unstable manifolds, intersect every neighborhood of either equilibrium $(\pm 1, 0)$ yielding a rather complicated mix of dynamics. However, points lying on the degenerate hyperbola for $\kappa = \pm 1$ do converge to the equilibrium $(\pm 1, 0)$, except for a countable set of points which escape in finite time.

In n dimensions, the n-folds (5.9) are defined for every value of $\kappa_1, \dots, \kappa_n$. Moreover, setting $\kappa_n = \kappa_{n-1} = \cdots = \kappa_2 = 0$ we obtain an invariant subset of $\mathbb{R}^{2n}$ on which the n-dimensional algorithm restricts to the first-order algorithm on the hyperbola defined by $\kappa = \kappa_1$. Therefore, in addition to the equilibrium structure described in Section 5 and the convergence analysis in Section 6 yielding a parameterization of the global stable manifolds of these equilibria, we also know (see [5])

Proposition 8.1. *For any $p \geq 3$, there exist infinitely many periodic points of period p for the fast filtering algorithm. Arbitrarily close to any one of these initial conditions is an initial condition which generates an unbounded orbit. In particular, in the sign indefinite region (in which trajectories cannot converge to equilibria) the fast filtering algorithms can exhibit sensitivity to initial conditions.*

We refer the reader to [5] for further details of the various kinds of asymptotic behavior in the case $n = 1$.

In the case $n = 2$, the fast filtering algorithm (4.4) takes the form

$$\alpha_1(t+1) = \frac{1}{1-\gamma_{t+1}^2}\alpha_1(t) + \frac{\gamma_{t+1}}{1-\gamma_{t+1}^2}\frac{\gamma_t}{1-\gamma_t^2}\alpha_2(t) \tag{8.5a}$$

$$\alpha_2(t+1) = \frac{\alpha_2(t)}{1-\gamma_t^2} \tag{8.5b}$$

$$\gamma_{t+1} = -\alpha_1(t)\gamma_t - \alpha_2(t)\gamma_{t-1} \tag{8.5c}$$

and the invariant manifold X_D becomes

$$\begin{cases} 2(r_1\alpha_1 + \alpha_1\alpha_2 + \gamma_0\gamma_1\alpha_2) = \kappa_1(\alpha_2^2 + r_1\alpha_1^2 + r_2) \\ \qquad\qquad\qquad\qquad\quad 2\alpha_2 = \kappa_2(\alpha_2^2 + r_1\alpha_1^2 + r_2) \end{cases} \tag{8.6}$$

where $r_1 = 1 - \gamma_0^2$ and $r_2 = (1-\gamma_0^2)(1-\gamma_1^2)$ are defined as in (3.7), as long as d_0, given by (4.11), is nonzero. The other cases are covered by dividing one or both of the equations in (8.6) by κ_1 and κ_2 respectively and allowing these constants to take infinite values.

In the second-order case the invariant manifold X_D may have a singular point not only if $a(z)$ and $b(z)$ have a root $z = 1$ or $z = -1$ in common, which may occur outside of the equilibrium set, but also if

$$a(z) = b(z) = (z+\lambda)(z+1/\lambda)$$

which can only occur in an equilibrium point, because $\gamma_0 = \gamma_1 = 0$ in this case. By Lemma 5.1, equilibria are precisely the points of the form $(\alpha_\infty, 0)$. Inserting $(\alpha_\infty, 0)$ in (8.6) yields the constants κ_1, κ_2 defining the invariant manifold containing this point. In fact,

$$\begin{cases} \kappa_1 = \dfrac{2\alpha_{\infty 1}(1+\alpha_{\infty 2})}{\alpha_{\infty 2}^2 + \alpha_{\infty 1}^2 + 1} \\ \kappa_2 = \dfrac{2\alpha_{\infty 2}}{\alpha_{\infty 2}^2 + \alpha_{\infty 1}^2 + 1} \end{cases} \tag{8.7}$$

Conversely, it follows from Theorem 5.9 and Theorem 7.1 that to each point (κ_1, κ_2) such that $D(z, z^{-1})$ is sign definite, there corresponds a unique α_∞, such that $\alpha_\infty(z)$ is stable, i.e., all its zeros lie inside the unit circle. These α_∞ are precisely the points in the closed triangular stability region depicted in Figure 8.3

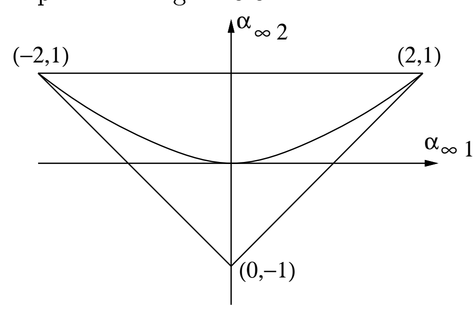

Figure 8.3

Since the roots of

$$D(z, z^{-1}) = r_\infty \alpha_\infty(z)\alpha_\infty(1/z) \tag{8.8}$$

are the eigenvalues of the symplectic matrix S (Proposition 6.6), each point in the closed triangle depicted in Figure 8.3 corresponds to a particular eigenvalue configuration for which there is convergence. Excluding the segment $\alpha_{\infty 2} = 0$ which corresponds to the case $n = 1$, the points in the interior of the triangle correspond to the situations when there are no eigenvalues on the unit circle. Below the parabola $\alpha_{\infty 2} = \alpha_{\infty 1}^2/4$, there are four real eigenvalues, while above there are two complex pairs. On the boundary of the triangle there are eigenvalues on the unit circle, but they are always of even multiplicity, as a simple application of (8.8) shows. The rest of the plane, outside of the triangle, corresponds to unstable solutions of the polynomial factorization problem (8.8) and hence to unstable equilibria $(\alpha_\infty, 0)$. To each point below the parabola in the interior of the triangle there corresponds one strictly unstable and two saddle equilibria outside the triangle. For an interior point above the parabola there is only one equilibrium outside the triangle and it is strictly unstable.

For all points in the interior of the triangle the invariant manifold X_D defined by (8.6) is a smooth surface. As shown in Section 6, not only is the invariant manifold through such a point locally a stable manifold, but actually it constitutes a global stable manifold, excluding the unstable equilibria, their stable manifolds, and the measure zero set of points which escape in finite time. Using the same argument as in the first-order case, $E = \{(\alpha_\infty, 0) : \alpha_\infty \in \mathbb{R}^2\}$ is a global center manifold through $(\alpha_\infty, 0)$ of dimension 2, for any α_∞ which does not lie on the lines through $(-2,1)$, $(2,1)$ and $(0,-1)$, i.e. on the boundary of the triangle.

The points on the boundary of the triangle are all singular. In fact, the invariant manifolds corresponding to the points on the line segment between $(-2,1,0,0)$ and $(2,1,0,0)$ as well as that of the point $(0,-1,0,0)$ have dimensions less than two. The center manifolds containing these points all have dimension four, while the center manifolds containing the points on the open boundary segments extending from the corner $(0,-1,0,0)$ have dimension three.

An initial condition $(\alpha, \gamma) \in \mathbb{R}^4$ for the fast filtering algorithm which does not belong to the plane $(\alpha_\infty, 0)$ of equilibria may or may not converge to an equilibrium. Figure 8.4 shows the plane $\alpha \mapsto (\alpha, \gamma)$ where γ is fixed so that, in this example, $\gamma_0 = 1/2$ and $\gamma_1 = 1/3$.

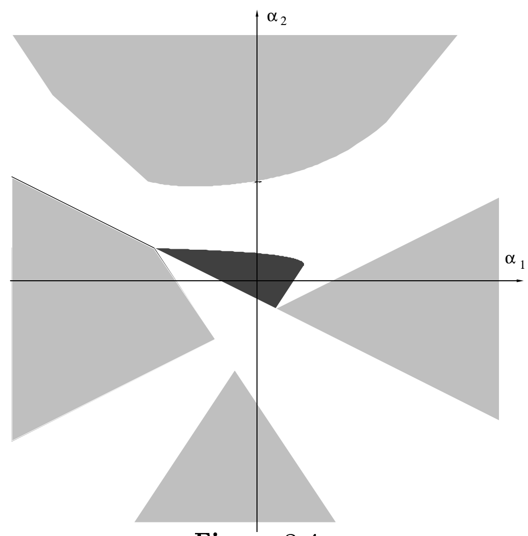

Figure 8.4

Each point in the bounded shaded region in Figure 8.4 corresponds to a positive real function $v(z)$, and hence to a *bona fide* stochastic system, and converges, by classical results, to a stable equilibrium $(\alpha_\infty, 0)$ in the triangle of Figure 8.3. This is precisely the solution set of the rational covariance extension problem for which the covariance data $\{c_1, c_2\}$ is prescribed so that the Schur parameters are $\gamma_0 = 1/2$ and $\gamma_1 = 1/3$. Initial conditions in the four unbounded shaded regions also correspond to orbits which converge to stable or unstable equilibria $(\alpha_\infty, 0)$ except for a zero measure set which escape in finite time.

As an example we may now choose the point $(0, 2, 1/2, 1/3)$, which lies in the topmost shaded unbounded region. We see from the simulation below that, using this point as an initial condition, the fast filtering algorithm (8.5) converges after having violated the positive real condition $|\gamma_t| < 1$ twice, showing that the corresponding $v(z)$ is not positive real. However, after six steps the iterate is inside the bounded positive real region and will remain there.

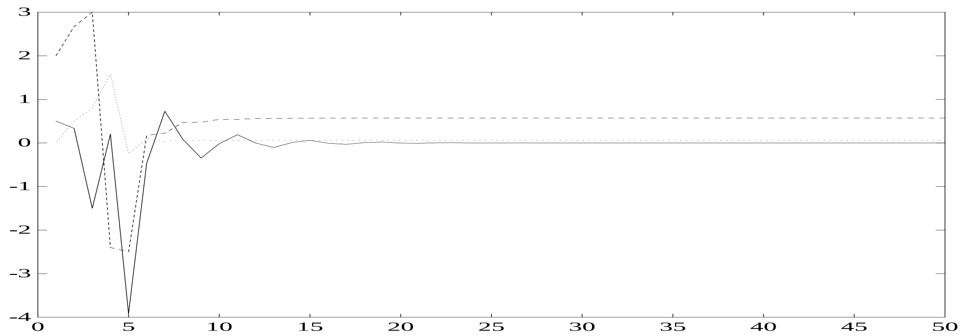

Figure 8.5: Plot of α_1 (dotted line),α_2 (dashed line),γ (solid line)

What happens if the initial condition (α, γ) lies in the white region of Figure 8.4? These points correspond to sign indefinite $D(z, z^{-1})$ and according to Proposition 8.1 we have at least three kinds of behavior.

(i) (α, γ) is a periodic point;
(ii) the orbit of (α, γ) is dense on some manifold;
(iii) there is finite–time escape.

These nonconvergent initial conditions and the invariant manifolds on which they lie correspond to the situations when S has eigenvalues of odd multiplicity on the unit circle. In the literature there has been a tendency to exclude the case with eigenvalues on the unit circle, as being a rather complicated nongeneric case, but, as our analysis shows, this situation actually corresponds to an open unbounded set of initial conditions. Cases (i) and (iii), however, occur only for a measure zero subset of the white region. We refer the reader to [6] for simulations illustrating these types of dynamical behavior. Here we show only one simulation which illustrates that in the white region the fast filtering algorithm is extremely sensitive to the initial conditions. Consider the periodic point of period 144 corresponding to $\beta_1 = \sec \pi/8$ and $\beta_2 = \sec \pi/9$, where

$$\frac{2}{\beta_1} = \frac{-\kappa_1 + \sqrt{\kappa_1^2 - 8\kappa_2 + 8\kappa_2^2}}{2\kappa_2},$$
$$\frac{2}{\beta_2} = \frac{-\kappa_1 - \sqrt{\kappa_1^2 - 8\kappa_2 + 8\kappa_2^2}}{2\kappa_2}$$

If we round off α_0 to the fifteenth digit, we obtain an orbit which is dense on the invariant manifold and part of whose γ-trajectory is depicted in Figure 8.6. This dynamical behavior is apparently quite different to that of the periodic point which the new initial condition approximates.

If S has all its eigenvalues on the unit circle but at least two of them are real, then the dynamics degenerates to the first-order case. The interesting fact here is that $a(z)$ and $b(z)$ having a common pair of reciprocal roots which are not 1 corresponds to an equilibrium lying on the line $(\alpha_{\infty,1}, 1, 0, 0)$ where $|\alpha_{\infty,1}| > 2$, i.e. the part of the line which is not in the boundary of the triangle of Figure 8.3. Moreover, this equilibrium is a saddle point.

A complete description of the positive real, sign definite and sign indefinite regions is available in the case $n = 2$, as reported in [19] where also many simulation results are given. Earlier graphical simulations of the positive real region, in the case $n = 2$, are contained in T. T. Georgiou's thesis [15]. Curiously, all graphical representations of the positive real region $\mathcal{A}_+(n)$ for γ fixed of which we are aware seem to be convex. Convexity of $\mathcal{A}_+(n)$ would in fact imply a Kharitonov-like property, viz. star-shapedness about the maximum-entropy filter, conjectured and established for the case $n = 1$ by Kimura. In this direction it is known that for reasons concerning the geometry of the spaces of real and of complex Schur polynomials, the convexity $\mathcal{A}_+(2)$ seems to be decidedly nontrivial. In general, although examples show [4] that $\mathcal{A}_+(n)$ can fail to be star-shaped for $n \geq 3$, $\mathcal{A}_+(n)$ is in fact always a Euclidian space [4].

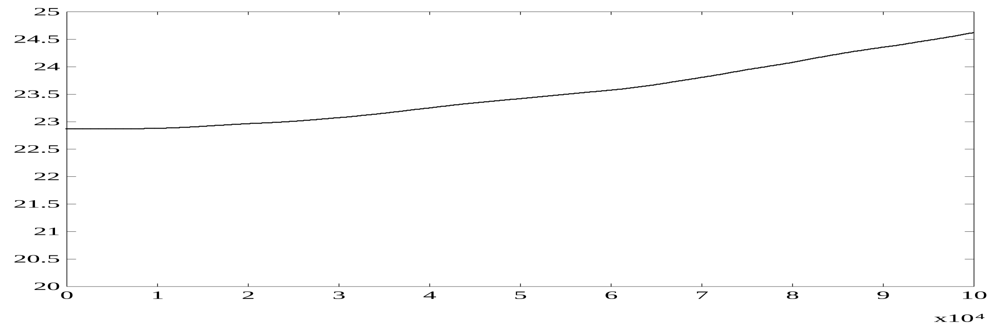

Figure 8.6

Appendix

In this section, we provide the proofs deferred from Sections 4 and 6.

Proof of Theorem 4.3. Under the map $\mathcal{F}^{-1}$ of Corollary 3.4 the initial conditions (a, g) are transformed to (α, γ) where $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$ are the first n Schur parameters of $v(z)$ and $\alpha_1, \alpha_2, \dots, \alpha_n$ are the parameters in the Kimura-Georgiou parameterization (3.9). Under the same map, $(a(t), g(t))$ goes into $(\alpha(t), \gamma(t))$ which, according to Corollary 3.4, satisfies

$$a(t) = \varphi_n(\gamma(t)) + \Phi_n(\gamma(t))\alpha(t), \tag{A-1a}$$

$$b(t) = \psi_n(\gamma(t)) + \Psi_n(\gamma(t))\alpha(t). \tag{A-1b}$$

$$g(t) = \frac{1}{2}[b(t) - a(t)] \tag{A-1c}$$

Now, by Lemma 4.2, $\{\gamma_t, \gamma_{t+1}, \gamma_{t+2}, \dots\}$ is the Schur parameter sequence of $v_t(z)$, and consequently (4.7) must hold. Therefore, if we can prove (4.8), then we have shown that (4.4b) holds. To this end note that, in view of (4.7), the last of equations (A-1a) reads

$$a_n(t+1) = -\gamma_{t+n-1} - \gamma_{t+n-2}\alpha_1(t+1) - \cdots - \gamma_{t+1}\alpha_{n-1}(t+1) + \alpha_n(t+1)$$

so if we can prove that

$$a_n(t+1) = (1 + \gamma_t)\alpha_n(t+1), \tag{A-2}$$

then (4.8) follows. However, from (A-1) we see that

$$\alpha_n(t) = a_n(t) + g_n(t) \tag{A-3}$$

for all $t = 0, 1, 2, \dots$, and then (A-2) follows from the bottom equations in each of (2.16a) and (2.16b). This establishes (4.4b).

To prove (4.4a) first note that since the dynamical system is time-invariant it is enough to show that $\alpha(1) = A(\gamma)\alpha$, i.e. that

$$\alpha = A(\gamma)^{-1}\alpha(1) \tag{A-4}$$

where

$$A(\gamma)^{-1} = \begin{bmatrix} 1-\gamma_{n-1}^2 & -\gamma_{n-1}\gamma_{n-2} & -\gamma_{n-1}\gamma_{n-3} & \cdots & -\gamma_{n-1}\gamma_0 \\ 0 & 1-\gamma_{n-2}^2 & -\gamma_{n-2}\gamma_{n-3} & \cdots & -\gamma_{n-2}\gamma_0 \\ 0 & 0 & 1-\gamma_{n-3}^2 & \cdots & -\gamma_{n-3}\gamma_0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1-\gamma_0^2 \end{bmatrix} \tag{A-5}$$

In view of (4.8), proving (A-4) amounts to proving that

$$\alpha_j = \gamma_{n-j}\gamma_n + \gamma_{n-j}\gamma_{n-1}\alpha_1(1) + \cdots + \gamma_{n-j}\gamma_{n-(j-1)}\alpha_{j-1}(1) + \alpha_j(1), \tag{A-6}$$

for $j = 1, 2, \ldots, n$. Now, after the change of coordinates (A-1), the k-th equation of (2.16a) reads

$$\begin{aligned} &\varphi_{nk}^{(1)} + \varphi_{n-1,k-1}^{(1)}\alpha_1(1) + \cdots + \varphi_{n-k+1,1}^{(1)}\alpha_{k-1}(1) + \alpha_k(1) \\ &= \frac{1}{1-\gamma_0}\{\pi_{nk} - \rho_{n,k+1} + (\pi_{n-1,k-1} - \rho_{n-1,k})\alpha_1 + \cdots + (\pi_{n-(k-1),1} - \rho_{n-(k-1),2})\alpha_{k-1}\} + \alpha_k \end{aligned} \tag{A-7}$$

where $\varphi_{tk}^{(1)} := \varphi_{tk}(\gamma(1))$, and ρ_{tk} and π_{tk} are the coefficients of polynomials (3.16) and (3.18) respectively. Note that $\{\varphi_{tk}^{(1)}\}$ are the coefficients of the Szegö polynomials $\{\varphi_t^{(1)}(z)\}$ corresponding to the shifted Schur parameter sequence $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ which are related to $\{\varphi_t(z)\}$ through the algebraic identity [16] (see also [12, 9])

$$\begin{bmatrix} \psi_{t+1}(z) \\ \varphi_{t+1}(z) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (1+\gamma_0)(z+1) & (1-\gamma_0)(z-1) \\ (1+\gamma_0)(z-1) & (1-\gamma_0)(z+1) \end{bmatrix}\begin{bmatrix} \psi_t^{(1)}(z) \\ \varphi_t^{(1)}(z) \end{bmatrix},$$

which can be inverted to yield

$$\begin{bmatrix} \psi_t^{(1)}(z) \\ \varphi_t^{(1)}(z) \end{bmatrix} = \frac{1}{2}\frac{1}{z(1-\gamma_0^2)}\begin{bmatrix} (1-\gamma_0)(1+z) & (1-\gamma_0)(1-z) \\ (1+\gamma_0)(1-z) & (1+\gamma_0)(1+z) \end{bmatrix}\begin{bmatrix} \psi_{t+1}(z) \\ \varphi_{t+1}(z) \end{bmatrix}.$$

Then

$$\varphi_t^{(1)}(z) = \frac{1}{2(1-\gamma_0)}[\pi_{t+1}(z) - z\rho_{t+1}(z)] \tag{A-8}$$

Using recursion (3.5) it is easy to see that

$$\varphi_{t+1}(z) = z\varphi_t(z) + \gamma_t\gamma_{t-1}z\varphi_{t-1}(z) + \gamma_t\gamma_{t-2}z\varphi_{t-2}(z) + \cdots + \gamma_t\gamma_0 z - \gamma_t. \tag{A-9}$$

Similarly, changing the signs of the Schur parameters in (A-9), we also have

$$\psi_{t+1}(z) = z\psi_t(z) + \gamma_t\gamma_{t-1}z\psi_{t-1}(z) + \gamma_t\gamma_{t-2}z\psi_{t-2}(z) + \cdots + \gamma_t\gamma_0 z + \gamma_t. \tag{A-10}$$

Combining (A-9) with (A-10) yields the recursions

$$\pi_{t+1}(z) = z\{\pi_t(z) + \gamma_t\gamma_{t-1}\pi_{t-1}(z) + \cdots + \gamma_t\gamma_1\pi_1(z) + \gamma_t\gamma_0\}, \tag{A-11}$$

and

$$\rho_{t+1}(z) = z\rho_t(z) + \gamma_t\gamma_{t-1}z\rho_{t-1}(z) + \cdots + \gamma_t\gamma_1 z\rho_1(z) + \gamma_t. \tag{A-12}$$

Now, inserting (A-11) and (A-12) into (A-8), we obtain

$$\begin{aligned} \varphi_t^{(1)}(z) = \frac{1}{1-\gamma_0}\{&\pi_t(z) - z\rho_t(z) + \gamma_t\gamma_{t-1}[\pi_{t-1}(z) - z\rho_{t-1}(z)] \\ &+ \gamma_t\gamma_{t-2}[\pi_{t-2}(z) - z\rho_{t-2}(z)] + \cdots + \gamma_t\gamma_1[\pi_1(z) - z\rho_1(z)]\} - \gamma_t \end{aligned}$$

which, after identifying coefficients and observing that $\rho_{j1} = \gamma_0$ for $j = 1, 2, 3, \dots$ yields

$$\varphi_{tk}^{(1)} = \frac{1}{1-\gamma_0}[\pi_{tk} - \rho_{t,k+1} + \gamma_t\gamma_{t-1}(\pi_{t-1,k-1} - \rho_{t-1,k}) + \cdots$$
$$+ \gamma_t\gamma_{t-k+1}(\pi_{t-k+1,1} - \rho_{t-k+1,2})] + \gamma_t\gamma_{t-k}, \quad k = 1, \dots, n. \tag{A-13}$$

We now prove (A-6) by induction. For $j = 1$, (A-7) reads

$$\varphi_{n1}^{(1)} + \alpha_1(1) = \frac{1}{1-\gamma_0}[\pi_{n1} - \rho_{n2}] + \alpha_1. \tag{A-14}$$

On the other hand, (A-13) yields

$$\varphi_{n1}^{(1)} = \frac{1}{1-\gamma_0}[\pi_{n1} - \rho_{n2}] + \gamma_n\gamma_{n-1}$$

and therefore

$$\alpha_1 = \alpha_1(1) + \gamma_n\gamma_{n-1},$$

which shows that (A-6) is true for $j = 1$. Next, suppose that (A-6) holds for $j = 1, 2, \dots, k-1$. We need to prove that (A-6) holds for $j = k$. To this end, use (A-6) for $j = 1, 2, \dots, k-1$ to eliminate $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$ from (A-7) and use (A-13) to eliminate $\varphi_{nk}^{(1)}, \varphi_{n-1,k-1}^{(1)}, \dots, \varphi_{n-k+1,1}^{(1)}$. This yields, after some simple calculations, (A-6) for $j = k$ as required.

Finally, formula (4.9) is obtained from (2.17) by merely inserting $a(t)$ and $g(t)$ as exhibited in (A-1). □

Proof of Lemma 4.4. It follows from (3.5) that $\{\pi_t\}$ as defined by (3.18) satisfies the recursion

$$\begin{cases} \pi_{t+1}(z) = z\pi_t(z) + \gamma_t\rho_t^*(z) & \pi_0(z) = 1 \\ \rho_{t+1}^*(z) = \rho_t^*(z) + \gamma_t z\pi_t(z) & \rho_0^*(z) = 0 \end{cases}$$

where $\rho_t^*(z) := z^n\rho_t(1/z)$, from which (4.12) is easily derived. Next, let $D(z, z^{-1})$ be defined by (2.13), and let $d(z)$ be the corresponding polynomial in (4.10). In view of (3.12)

$$D(z, z^{-1}) = \sum_{i=0}^{n}\sum_{j=0}^{n} \alpha_i\alpha_j\sigma_{ij}(z, z^{-1}); \tag{A-15}$$

where $\alpha_0 = 1$ and

$$\sigma_{ij}(z, z^{-1}) := \frac{1}{2}[\varphi_i(z)\psi_j(1/z) + \psi_i(z)\varphi_j(1/z)]. \tag{A-16}$$

Now, it is well-known and easy to check that

$$\sigma_{ii}(z, z^{-1}) = r_i, \tag{A-17a}$$

and hence by using (3.5) we see that

$$\sigma_{i,i-1}(z, z^{-1}) = r_{i-1}z. \tag{A-17b}$$

Then, for $j = 1, \dots, i$, we obtain, by induction and repeated use of (3.5)

$$\sigma_{i,i-j}(z, z^{-1}) = r_{i-j}(z^j + p_1 z^{j-1} + \cdots + p_{j-1}z), \tag{A-17c}$$

where $p_1, p_2, \dots, p_{j-1}$ are functions of $\gamma_{i-j}, \gamma_{i-j+1}, \dots, \gamma_{i-1}$ only. Consequently, it follows from (A-15) that

$$d_0 = \alpha_n^2 + r_1\alpha_{n-1}^2 + \cdots + r_n, \tag{A-18}$$

and that

$$d(z) - \frac{1}{2}d_0 = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \alpha_i\alpha_j\sigma_{n-i,n-j}(z, z^{-1})$$
$$= \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \alpha_i\alpha_j\sigma_{n-i,n-j}(z, z^{-1}) + \alpha_n \sum_{j=1}^{n} \alpha_{n-j}\pi_j(z) \tag{A-19}$$

because $\sigma_{j0} = \pi_j$. Now, writing $d^{(n)}$ instead of $d(z)$ to stress the fact that n is the dimension of α or γ (but *not* necessarily the degree of $d(z)$), we observe that the first term of (A-19) equals $d^{(n-1)}(z) - \frac{1}{2}d_0$ except that σ_{kl} has been replaced by $\sigma_{k+1,l+1}$, which replacement according to (A-17) amounts to exchanging $\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$ by $\{\gamma_1, \gamma_2, \dots, \gamma_n, \gamma_1, \gamma_2, \dots, \gamma_n\}$. Consequently, since $d^{(1)}(z) = \frac{1}{2}d_0 + \alpha_1 z$, the coefficients of (4.10) are generated by the recursion in $d^{(k)}$ defined in the lemma.

Finally, suppose $d_i = 0$ for $i = 1, 2, \dots, n$. Then, since $1 - \gamma_i^2 \neq 0$ for $i = 0, 1, 2, \dots, n-1$, it follows from the recursion in $d^{(k)}$ that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. But then, by (4.11), $d_0 = r_n := \prod_{i=0}^{n-1}(1 - \gamma_i^2) \neq 0$. Hence at least one of the coefficients must be nonzero as claimed. □

Proof of Lemma 6.8. Let $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ be a matrix basis of $\mathcal{U}$ as in (6.19). Then

$$\mathcal{W} = \left\{ \begin{bmatrix} X \\ Y \end{bmatrix} z \mid z \in \ker Y \right\} \tag{A-20}$$

The S-invariance of $\mathcal{U}$ implies that there is a $n \times n$ matrix R such that

$$S \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R \tag{A-21}$$

and therefore (i) holds if and only if $Rz \in \ker Y$ for all $z \in \ker Y$, i.e. $\ker Y$ is R-invariant. Set $z \in \ker Y$. Then, using (6.14), the second block of (A-21) yields

$$gg'(A')^{-1}Xz = YRz \tag{A-22}$$

Since $\mathcal{U}$ is Lagrangian, $X'Y = Y'X$ so that $z'X'Y = 0$, and therefore

$$[z'X'g][g'(A')^{-1}Xz] = 0 \tag{A-23}$$

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be the largest subspaces of $\ker Y$ for which $g'(A')^{-1}Xz = 0$ and $g'Xz = 0$ respectively. Then for (A-23) to hold for all $z \in \ker Y$ either $\mathcal{V}_1$ or $\mathcal{V}_2$ must be all of $\ker Y$. In fact, if there are two one-dimensional subspaces $l_1 \in \mathcal{V}_1$ and $l_2 \in \mathcal{V}_2$, then an arbitrary point in the plane spanned by l_1 and l_2 must belong to $\ker Y$ and hence to either $\mathcal{V}_1$ or $\mathcal{V}_2$ for (A-23) to hold; say $\mathcal{V}_1$. But then the whole plane must belong to $\mathcal{V}_1$, and hence also l_2.

Now, first suppose that $g'(A')^{-1}Xz = 0$ for all $z \in \ker Y$. Then, it follows from (A-22) that $\ker Y$ is R-invariant, and hence (i) holds.

Next, suppose that $g'Xz = 0$ for all $z \in \ker Y$. Since $\alpha_n \neq 0$, $\kappa_n \neq 0$, and hence S is nonsingular (Proposition 6.6). Therefore, in view of (A-21), R is also nonsingular so that

$$S^{-1} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R^{-1} \tag{A-24}$$

and consequently

$$S^{-1}\mathcal{W} = \left\{ \begin{bmatrix} X \\ Y \end{bmatrix} R^{-1}z \mid z \in \ker Y \right\}$$

which belongs to $\mathcal{W}$ if and only if $R^{-1}z \in \ker Y$ for all $z \in \ker Y$. Now, by (6.12), $S^{-1} = \hat{J}^{-1}S'\hat{J}$ and hence (A-24) is equivalent to

$$S' \begin{bmatrix} -Y \\ X \end{bmatrix} = \begin{bmatrix} -Y \\ X \end{bmatrix} R^{-1} \tag{A-25}$$

Taking $z \in \ker Y$ and remembering that $g'Xz = 0$, the top block of (A-25) yields $YR^{-1}z = 0$ which is what is required for condition (ii) to hold. Hence, we have proved that at least one of conditions (i) and (ii) holds.

Finally, we shall prove that these conditions are actually equivalent. Suppose $\dim \mathcal{W} = k$. If $k = 0$, the statement is trivial, so we assume that $k > 0$. Then

$$\mathcal{W} = \operatorname{Im} \begin{bmatrix} U \\ 0 \end{bmatrix} \tag{A-26}$$

for some full-rank $n \times k$ matrix U such that

$$S \begin{bmatrix} U \\ 0 \end{bmatrix} = \begin{bmatrix} U \\ 0 \end{bmatrix} T \tag{A-27}$$

Since S is nonsingular, so is T, so (A-27) is equivalent to

$$S^{-1} \begin{bmatrix} U \\ 0 \end{bmatrix} = \begin{bmatrix} U \\ 0 \end{bmatrix} T^{-1} \tag{A-28}$$

which holds if and only if $S^{-1}\mathcal{W} \subset \mathcal{W}$. This concludes the first part of the lemma. The proof of the second part, concerning $\tilde{\mathcal{W}}$ is analogous. ☐

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