OPTIMAL DAMPING OF FORCED OSCILLATIONS IN DISCRETE-TIME SYSTEMS*

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Abstract. In this paper we consider a linear discrete-time control system affected by an additive sinusoidal disturbance with known frequencies but unknown amplitudes and phases. The problem is to damp this forced oscillation in an optimal fashion. We show that the natural solution from the point of view of optimal control is neither robust with respect to errors in the frequencies, and thus not optimal in practice, nor independent of the unknown amplitudes and phases. The main result of this paper concerns the existence and design of a realizable, robust optimal regulator which is universal in the sense that it does not depend on the unknown amplitudes and phases and is optimal for all choices of such parameters. The regulator allows for a considerable degree of design freedom to satisfy other design specifications. Finally, it is shown that this regulator is optimal also for a wide class of stochastic control problems.

1. Introduction

Consider the linear discrete-time system

$$x_{t+1} = Ax_t + Bu_t + Cw_t, \quad x_0 = a \tag{1.1}$$

where $\{x_t\}$ is an n-dimensional real state sequence, $\{u_t\}$ is a k-dimensional real control sequence,

$$w_t = \begin{bmatrix} \alpha_1 \cos(\omega_1 t + \varphi_1) \\ \alpha_2 \cos(\omega_2 t + \varphi_2) \\ \vdots \\ \alpha_\nu \cos(\omega_\nu t + \varphi_\nu) \end{bmatrix} \tag{1.2}$$

is an ν-dimensional real sinusoidal disturbance with known frequencies

$$-\pi < \omega_1 < \omega_2 < \cdots < \omega_\nu \leq \pi \tag{1.3}$$

but unknown amplitudes $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ and phases $\varphi_1, \varphi_2, \ldots, \varphi_\nu$, and A, B, C are given real matrices of appropriate dimensions so that (A, B) is a stabilizable pair and C has no trivial (i.e., zero) columns.

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One application area of interest which can be modeled by such equations is that of flight control through wind shear, where the sinusoidal forcing terms arise from a model for wind shear based on harmonic oscillations [24, 18]. Various criteria such as set point control of the clime rate or minimization of other performance criteria have been proposed in the literature [24, 18, 32]. Other applications include vibration damping for industrial machines, noise reduction in vehicles and transformers, periodic disturbance reduction in disk drives, and the control of the roll motion of a ship (see, e.g., [4, 8, 9, 11, 12, 14, 27, 29, 30]).

Another possible criterion for these problems is to force some output signal $y_t = Lx_t$ to tend asymptotically to zero. Since the sinusoidal disturbance can be modeled as a critically stable "exosystem", a discrete-time version of the methods proposed in [6, 7] could be used for this purpose. However, such solutions are not always available, as some rather strict geometric conditions need to be satisfied.

In this paper we shall consider minimization of a quadratic performance measure which reflects the ability of damping the steady-state solution of (1.1) produced by the sinusoidal disturbance. More precisely, the control objective is to minimize the cost functional

$$\Phi = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \Lambda(x_t, u_t), \tag{1.4}$$

where $\Lambda(x, u)$ is the real quadratic form

$$\Lambda(x, u) = \begin{pmatrix} x \\ u \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{1.5}$$

satisfying the frequency-domain condition (1.13) below and with Q and R symmetric, i.e. $Q = Q^*$ and $R = R^*$. This cost function is appropriate for most of the applications mentioned above. However, in many problems of noise reduction or vibration attenuation in vehicles, especially in helicopters, the harmonic disturbance needs to enter the cost function in a quadratic manner in order to allow some system output to track a harmonic reference signal of type (1.2); see, e.g., [23, 13]. This situation is not covered by our present formulation but is considered in a sequel to this paper [22].

The mathematical problem under consideration in this paper is to find among all regulators

$$u_t = f(t, x_t, x_{t-1}, \dots, x_0), \tag{1.6}$$

which are stabilizing in the sense that they generate a state process x_t satisfying the admissibility condition

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} |x_t| = 0 \tag{1.7}$$

for each choice of disturbance (1.2), one that minimizes the cost functional Φ. We would like to find an optimal regulator with the following special properties. It is realizable in the sense that it has a bounded finite memory

$$u_t = \hat{f}(x_t, x_{t-1}, \dots, x_{t-\tau}, u_{t-1}, \dots, u_{t-\tau}) \quad \text{for some } \tau \tag{1.8}$$

and does not depend on the unknown parameters $\alpha_1, \alpha_2, \dots, \alpha_\nu$ and $\varphi_1, \varphi_2, \dots, \varphi_\nu$. More precisely the function $\hat{f}$ corresponding to the optimal controller should not depend on the amplitudes and phases while of course the optimal process (x_t, u_t) and the cost function Φ certainly do depend on these parameters. In other words, we want to find a regulator (1.8) which is *universal* in the sense that it solves the complete family of optimization problems corresponding to different choices of $\alpha_1, \alpha_2, \dots, \alpha_\nu$ and $\varphi_1, \varphi_2, \dots, \varphi_\nu$. Moreover, the optimal regulator must be *robust* with respect to the known frequencies $\omega_1, \omega_2, \dots, \omega_\nu$ in the following (nonstandard) sense: Since, in practice, the regulator will be computed from estimates $\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_\nu$ of the true frequencies $\omega_1, \omega_2, \dots, \omega_\nu$, the cost functional Φ must be continuous in the estimation errors $\omega_1 - \hat{\omega}_1, \omega_2 - \hat{\omega}_2, \dots, \omega_\nu - \hat{\omega}_\nu$ and tend to its true optimal value as the errors tend to zero. Otherwise, the regulator will not be optimal in practice. This formulation can be generalized to the situation of more general output feedback where some output and not the complete state is available for observation [21, 22].

We shall demonstrate that this problem has a solution in the class of linear regulators

$$D(\sigma)u_t = M(\sigma)x_t \tag{1.9}$$

for which the overall closed-loop system consisting of (1.1) and (1.9) is stable. Here σ is the forward shift operator $\sigma x_t = x_{t+1}$ and $D(\lambda)$ and $M(\lambda)$ are $k \times k$ and $k \times n$ matrix polynomials such that the leading coefficient of $D(\lambda)$ is nonsingular and $\deg M \leq \deg D$ so that $D(\lambda)^{-1}M(\lambda)$ is a proper rational matrix function. Of course, for such a regulator to be universal, the matrix polynomials $D(\lambda)$ and $M(\lambda)$ must not depend on the unknown amplitudes and phases.

Since therefore the optimal solutions belong to a class of linear stabilizing regulators, the admissibility condition (1.7) may seem unnecessarily weak. However, the point is that we want to prove optimality in the largest possible class of regulators, including nonlinear ones, and (1.7) turns out to be the natural stability condition for such a class.

The regulator (1.9) may also be written in the form

$$\begin{cases} z_{t+1} = Fz_t + Gx_t \\ u_t = Hz_t + Jx_t \end{cases}, \tag{1.10}$$

where F, G, H and J can be determined from the matrix fraction representation

$$D(\lambda)^{-1}M(\lambda) = H(\lambda I - F)^{-1}G + J \tag{1.11}$$

by means of some realization procedure. However, the matrix polynomials $D(\lambda)$ and $M(\lambda)$ need not be coprime, so for the sake of robustness it is practically more convenient to use the form (1.9). Also, if we replace the system (1.1) by

$$a(\sigma)y_t = b(\sigma)u_t + cw_t \tag{1.12}$$

for appropriately defined matrix polynomials $a(\lambda)$ and $b(\lambda)$, then we can reduce it to (1.1), but it also allows us consider many cases of systems (1.1) for which we can observe only an output $y_t = Lx_t$ where L is some matrix.

The quadratic form (1.5) could be indefinite but must satisfy the frequency-domain condition

$$\Lambda(\tilde{x}, \tilde{u}) \geq \delta(|\tilde{x}|^2 + |\tilde{u}|^2) \tag{1.13}$$

for some $\delta > 0$ and for all $\tilde{x} \in \mathbb{C}^n$, $\tilde{u} \in \mathbb{C}^k$, $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\lambda\tilde{x} = A\tilde{x} + B\tilde{u}$. This is a natural condition. In fact, it can be shown that if it fails in a strong way, i.e. there are $\tilde{x}$, $\tilde{u}$ and λ, $|\lambda| = 1$, such that $\Lambda(\tilde{x}, \tilde{u}) < 0$, then there is an external disturbance w_t such that $\inf \Phi = -\infty$ (see Appendix A). In the optimal damping problem the quadratic form Λ is usually of the type $\Lambda_c(x, u) = |c^*x|^2 + |u|^2$. If $\det(\lambda I - A) \neq 0$ for all $|\lambda| = 1$, then in this case the frequency condition (1.13) obviously holds. Here we allow for more general forms Λ, even such that are indefinite.

The outline of the paper is as follows. In Section 2 we present some preliminary optimality results for a general bounded external disturbance. In Section 3 we specialize this to harmonic disturbances, discuss some nonsolutions to the robust control problem, and reformulate the problem to be solved. In Section 4 we give a general characterization of the class of stabilizing regulators, which may be of interest in its own right. This parameterization turns out be related to, but not quite equivalent to, the Youla parameterization. Section 5 is devoted to the main result. Here we present a solution of the control problem stated above, and in Section 6 we illustrate this solution by a simple numerical example and some simulations. In Section 7 we show that this solution is also optimal when the disturbance w_t is generated by a certain harmonic stochastic system. Of course, if w_t is merely white noise or colored noise with known rational spectral density, the solution is well-known; see, e.g., [2, 5]. Our problem, however, actually corresponds to the case of colored noise but with *unknown* spectral density.

2. Preliminary optimality results

We recall the classical problem in control theory to minimize

$$\sum_{t=0}^{\infty} \Lambda(x_t, u_t) \tag{2.1}$$

when

$$x_{t+1} = Ax_t + Bu_t \tag{2.2}$$

and

$$|x_t| \in \ell_2, \qquad |u_t| \in \ell_2. \tag{2.3}$$

It is well-known that this problem has the optimal feedback solution

$$u_t = Kx_t, \tag{2.4}$$

where the gain

$$K = -(B^*PB + R)^{-1}(A^*PB + S)^* \tag{2.5}$$

is expressed in terms of the stabilizing solution of the matrix equation

$$P = A^*PA - (A^*PB + S)(B^*PB + R)^{-1}(A^*PB + S)^* + Q, \tag{2.6}$$

i.e. the symmetric solution P which, if it exists, renders the feedback matrix

$$\Gamma = A + BK \tag{2.7}$$

stable in the sense that all eigenvalues of Γ lie strictly inside the unit circle. (See, e.g., [16, 31, 25, 17] and articles in [3].) The matrix equation (2.6) is known as the *algebraic Riccati equation* or, originally and more correctly, the *Lur'e equation.*

The existence of a solution of (2.6) is equivalent to the existence of a Lyapunov function

$$V(x) = x^* P x \tag{2.8}$$

satisfying

$$V(Ax + Bu) - V(x) + \Lambda(x, u) = (u - Kx)^* \hat{R}(u - Kx) \tag{2.9}$$

for some matrices K and $\hat{R} = \hat{R}^* > 0$. This can be seen by merely forming the left member of (2.9) and completing squares, whereby (2.9) is obtained if and only if P satisfies (2.6). This procedure also shows that K must be given by (2.5) and that

$$\hat{R} = B^* P B + R. \tag{2.10}$$

We recall the following theorem which relates the frequency-domain condition introduced in Section 1 to the existence of an optimal solution to the problem to minimize (2.1) subject to (2.2) and (2.3) as well as to the existence of a stabilizing solution of (2.6). Different versions of this theorem can be found in [16, 31, 26, 25, 17], but the first result of this type was established by Kalman and Szegö [15] for the case $m = 1$. The case of infinite-dimensional systems were treated in [1, 19]. Using the results of [1, 19] all the results of this paper could be extended to the case that x_t, u_t are vectors in infinite dimensional Hilbert spaces.

Theorem 2.1. *Let (A, B) be stabilizable. Then the following statements are equivalent:*

(i) *there exist matrices $P = P^*$, $\hat{R} = \hat{R}^* > 0$ and K satisfying the identity* (2.9) *and rendering the matrix* (2.7) *stable;*
(ii) *for any initial condition $a \in \mathbb{R}^n$ there exists an optimal process (x_t, u_t) minimizing* (2.1) *subject to the constraints* (2.2) *and* (2.3)*;*
(iii) *the frequency-domain condition* (1.13) *holds.*

It is easy to see that (1.13) is an immediate consequence of (2.9) and the fact that Γ is stable and $\hat{R}$ is positive definite. Let us suppose for simplicity that A is a stable matrix. (The general case reduces to this one by the stabilizability of (A, B).) The relation (2.9) for real x, u implies that the same relation holds for complex x, u provided * denotes Hermitian conjugation. Taking $\tilde{x}$, $\tilde{u}$ and λ such that $\lambda\tilde{x} = A\tilde{x} + B\tilde{u}$ and $|\lambda| = 1$, (2.9) becomes

$$\Lambda(\tilde{x}, \tilde{u}) = (\tilde{u} - K\tilde{x})^* \hat{R}(\tilde{u} - K\tilde{x}) \geq 0$$

with equality if and only if $\tilde{u} = K\tilde{x}$, i.e. if and only if $(\lambda I - \Gamma)\tilde{x} = 0$. But, since Γ is a stable matrix, this is equivalent to $\tilde{x} = 0$ and hence $\tilde{u} = 0$. Since A has no eigenvalues on the unit circle, this establishes (1.13). The proof of the converse statement, namely that the the frequency domain condition (1.13) implies (i), is much harder.

We also remind the reader that the optimality of the control law (2.4) is immediate from (i). In fact, for any admissible process, (2.3) implies that $|x_t| \to 0$ as $t \to \infty$ and hence so does $V(x_t)$. Therefore (2.9) yields

$$\sum_{t=0}^{\infty} \Lambda(x_t, u_t) = V(x_0) + \sum_{t=0}^{\infty} (u_t - Kx_t)^* \hat{R}(u_t - Kx_t),$$

Since Γ is stable, the regulator (2.4) yields an admissible process, which is obviously optimal and uniquely defined by virtue of the fact that $\hat{R} > 0$.

Next we add a bounded external disturbance $\{v_t\}$ to the system (2.2) to obtain

$$x_{t+1} = Ax_t + Bu_t + v_t. \tag{2.11}$$

Then we must change both the stability condition (2.3) and the cost functional. In fact, we take

$$\Phi = \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \Lambda(x_t, u_t) \tag{2.12}$$

to be the cost functional to be minimized, and we say that the process (x_t, u_t) is *admissible* if it satisfies (2.11) and the stability condition (1.7), i.e.,

$$\lim_{t \to \infty} \frac{1}{\sqrt{t}} |x_t| = 0. \tag{2.13}$$

For simplicity, and with the obvious definition of the averaging operator $\mathrm{M}\{\cdot\}$, we shall also write

$$\Phi = \mathrm{M}\{\Lambda(x_t, u_t)\}. \tag{2.14}$$

A completion-of-squares argument such as the one above will still work but requires a more general Lyapunov function of the form

$$V(x, t) = x^* P x + 2{p_t}^* x + q_t, \tag{2.15}$$

where P is a stabilizing solution of (2.6). To this end we shall assume again that (A, B) is stabilizable and that the frequency-domain condition (1.13) holds so that such a stabilizing solution actually exists.

Lemma 2.2. *Let P be a stabilizing solution of* (2.6) *and suppose that the sequences $\{p_t\}$ and $\{q_t\}$ satisfy the recursions*

$$p_t = \Gamma^* p_{t+1} + \Gamma^* P v_t \tag{2.16}$$

and

$$q_{t+1} = q_t - v_t^* P v_t - 2p_{t+1}^* v_t + r_t^* \hat{R} r_t \tag{2.17}$$

for $t = 0, 1, 2, \dots,$ where Γ and $\hat{R}$ are defined by (2.7) *and* (2.10) *respectively, and*

$$r_t = -\hat{R}^{-1} B^* (p_{t+1} + P v_t). \tag{2.18}$$

Then the Lyapunov function (2.15) *satisfies*

$$V(x_{t+1}, t+1) - V(x_t, t) + \Lambda(x_t, u_t) = (u_t - Kx_t - r_t)^* \hat{R}(u_t - Kx_t - r_t) \tag{2.19}$$

along the trajectory of (2.11)*, where* K *is the gain* (2.5).

Proof. Using (2.9) and completing squares a straight-forward calculation shows that the left and right member of (2.19) differ by a term which is linear in x_t, whose coefficient is zero by (2.16), and a constant term, which is zero by virtue of (2.17). □

If $\det \Gamma \neq 0$ we have

$$p_{t+1} = (\Gamma^*)^{-1} p_t - P v_t \tag{2.20}$$

so that (2.17) and (2.18) can be replaced by

$$q_{t+1} = q_t + v_t^* P v_t - 2 p_t^* \Gamma^{-1} v_t + r_t^* \hat{R} r_t \tag{2.21}$$

respectively

$$r_t = -\hat{R}^{-1} B^* (\Gamma^*)^{-1} p_t. \tag{2.22}$$

However, since Γ is stable, (2.20) is strictly unstable in the forward direction. Moreover, (2.16), or (2.20), has a unique bounded solution, namely

$$p_t = \sum_{k=t}^{\infty} (\Gamma^*)^{k-t+1} P v_k. \tag{2.23}$$

It is easy to verify that (2.23) is true regardless of whether $\det \Gamma \neq 0$ or not, but if $\det \Gamma = 0$ the bounded solution of (2.16) is not unique.

Theorem 2.3. *Let* (A, B) *be stabilizable and suppose that the frequency-domain condition* (1.13) *holds so that* (2.6) *has a stabilizing solution* P*. Moreover, let* p_t *be the bounded solution* (2.23) *of* (2.16)*. Consider the problem to minimize the functional* (2.12) *subject to conditions* (2.11) *and* (2.13)*. Then the process* (x_t, u_t) *obtained by taking the control*

$$u_t = K x_t + r_t + \epsilon_t \tag{2.24}$$

in (2.11) *is optimal if* K *and* r_t *are given by* (2.5) *and* (2.22) *respectively and* $\{\epsilon_t\}$ *is a sequence such that*

$$\mathrm{M}\{|\epsilon_t|^2\} = 0. \tag{2.25}$$

The optimal value of the cost function is given by

$$\Phi_{min} = \mathrm{M}\{r_t^* \hat{R} r_t - 2 p_{t+1}^* v_t - v_t^* P v_t\}. \tag{2.26}$$

More specifically, for any admissible (x_t, u_t)*, the value of the cost functional is*

$$\Phi = \limsup_{T \to \infty} \{\frac{1}{T} \sum_{t=0}^{T} (u_t - K x_t - r_t)^* \hat{R} (u_t - K x_t - r_t) - \frac{1}{T} q_{T+1}\}. \tag{2.27}$$

If the limit $\lim_{T\to\infty} \frac{1}{T} q_{T+1}$ *exists, any optimal process* (x_t, u_t) *is produced by a controller* (2.24) *with* ϵ_t *satisfying* (2.25).

Proof. Set $V_t := V(x_t, t)$ and $\Lambda_t := \Lambda(x_t, u_t)$, where (x_t, u_t) is an admissible process. Then (2.19) yields

$$\frac{1}{T}\left[V_{T+1} - V_0\right] + \frac{1}{T}\sum_{t=0}^{T}\Lambda_t = \frac{1}{T}\sum_{t=0}^{T}(u_t - Kx_t - r_t)^* \hat{R}(u_t - Kx_t - r_t).$$

Since $|x_t^* P x_t| = o(t)$ and $|p_t^* x_t| = o(\sqrt{t})$ by admissibility condition (2.13),

$$\frac{1}{T}\left[V_{T+1} - V_0\right] = \frac{1}{T}q_{T+1} + o(1)$$

for any initial value q_0, and hence the cost functional (2.12) becomes (2.27). Since $\hat{R} > 0$, we obtain from (2.27) that

$$\Phi \geq \limsup_{T\to\infty}\left(-\frac{1}{T}q_{T+1}\right) \tag{2.28}$$

for any admissible control. Now, taking the control (2.24), the controlled system (2.11) becomes

$$x_{t+1} = \Gamma x_t + B(r_t + \epsilon_t) + v_t, \tag{2.29}$$

where, by construction, Γ is a stability matrix and r_t is bounded. Because ϵ_t satisfies (2.25), it is simple to show that the admissibility condition (2.13) is fulfilled (see Appendix B), and consequently (2.24) is an admissible control. Then we see from (2.28) and the condition $\hat{R} > 0$ that (2.24) is in fact optimal, and hence the minimum value of Φ is

$$\Phi_{\min} = \limsup_{T\to\infty}\left(-\frac{1}{T}q_{T+1}\right). \tag{2.30}$$

Using (2.21), we now transform (2.30) to (2.26). Conversely, suppose that (x_t, u_t) is optimal so that $\Phi = \Phi_{\min}$. Then, since the limit $\lim_{T\to\infty}\frac{1}{T}q_{T+1}$ exists,

$$\Phi = \Phi_{\min} + \mathrm{M}\{(u_t - Kx_t - r_t)^* \hat{R}(u_t - Kx_t - r_t)\}$$

implies that $\epsilon_t := u_t - Kx_t - r_t$ satisfies $\mathrm{M}\{\epsilon_t^* \hat{R}\epsilon_t\} = 0$. But $\hat{R} > 0$ and hence (2.25) follows. □

The control law described in this theorem is of course in general not satisfactory, because u_t depends through

$$r_t = -\hat{R}^{-1}B^*\sum_{k=t}^{\infty}(\Gamma^*)^{k-t}Pv_k \tag{2.31}$$

on future values of the disturbance v_k. Hence it is in general not realizable. As we shall see next, the objection disappears if v_t is harmonic, but new difficulties will appear.

Remark 2.4. All the results of this section remain valid when the disturbance v_t is allowed to be complex (while the other parameters remain real) provided that $p_t^* v_t$ and $p_{t+1}^* v_t$ are replaced by $\mathrm{Re}\{p_t^* v_t\}$ and $\mathrm{Re}\{p_{t+1}^* v_t\}$ respectively. Then p_t is complex while q_t remains real.

3. Optimal control when the external disturbance is harmonic

Let us now suppose that the external disturbance v_t in (2.11) is harmonic or, more precisely, that

$$v_t = Cw_t, \tag{3.1}$$

where

$$w_t = \begin{bmatrix} \beta_1 e^{i\theta_1 t} \\ \beta_2 e^{i\theta_2 t} \\ \vdots \\ \beta_m e^{i\theta_m t} \end{bmatrix}, \quad \beta_1, \beta_2, \dots, \beta_m \in \mathbb{C}, \tag{3.2}$$

and

$$-\pi < \theta_1 < \theta_2 < \cdots < \theta_m \leq \pi. \tag{3.3}$$

This allows us to write $w_t = D^t \beta$ where

$$D = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_m}) \quad \text{and} \quad \beta = \mathrm{col}\,(\beta_1, \beta_2, \dots, \beta_m). \tag{3.4}$$

This choice of v_t is more general than that in Section 1, and by redefining the matrix C it covers the systems description there.[1]

However, if the limits exist in the corresponding cost functions (1.4) as will be the case for the linear class (1.10), or (1.9), (see Theorem 4.4), the problem of Section 1 with a real disturbance, henceforth called the *real optimization problem*, can be embedded in the *complex optimization problem* with harmonic disturbance (3.2) and with the same (real) C as in Section 1. With the obvious modifications, described in Remark 2.4, Theorem 2.3 applies to this problem as well.

Proposition 3.1. *Let $m = \nu$, and set $\beta_j = \alpha_j e^{i\varphi_j}$ and $\theta_j = \omega_j$ for $j = 1, 2, \dots, m$. Then, if the process (x_t, u_t) is optimal for the complex optimization problem, the process $(Re\{x_t\}, Re\{u_t\})$ is optimal for the the real optimization problem, provided the limits in the cost functions* (1.4) *exist in both problems.*

Proof. Note that

$$\Lambda(x_t, u_t) = \Lambda(\mathrm{Re}\{x_t\}, \mathrm{Re}\{u_t\}) + \Lambda(\mathrm{Im}\{x_t\}, \mathrm{Im}\{u_t\})$$

and that $(\mathrm{Re}\{x_t\}, \mathrm{Re}\{u_t\})$ satisfies (1.1) with w_t given by (1.2) and $(\mathrm{Im}\{x_t\}, \mathrm{Im}\{u_t\})$ the same equation with cosine exchanged for sine and with $a = 0$. Therefore, if the corresponding limits in (1.4) exist, the complex optimization problem is decomposed into two real optimization problems, one of which is precisely that of Section 1. Clearly, the complex optimization problem is solved only if the two real ones are. □

[1]Let us denote C and w_t in (1.1) as C^0 and w_t^0 respectively. Then, if no ω_j equals 0 or π, $C = [C^0, C^0]$ and $C^0 w_t^0 = Cw_t$ in (3.1), and, in (3.2), $m = 2\nu$, $\beta_j = \frac{1}{2}\alpha_j e^{i\varphi_j}$, $\beta_{\nu+j} = \frac{1}{2}\alpha_j e^{-i\varphi_j}$, $\theta_j = \omega_j$ and $\theta_{\nu+j} = -\omega_j$ for $j = 1, 2, \dots, \nu$. If $\omega_k = 0$ or $\omega_k = \pi$, $\beta_k = \beta_{\nu+k}$ and we may take $\theta_k = \theta_{\nu+k}$, so the corresponding column in C^0 need be repeated in C. Note that C and Cw_t are real.

Let us now consider the optimization problem. For simplicity and to illustrate a point, let us, just for the moment, assume that $\det \Gamma \neq 0$ and let us take $\epsilon_t \equiv 0$ in Theorem 2.3. Then, by (2.22), $r_t = Ep_t$ where $E := -\hat{R}^{-1}B^*(\Gamma^*)^{-1}$, and therefore, in view of (2.29) and (2.20), the optimal process x_t, $u_t = Kx_t + Ep_t$ satisfies the system of equations

$$\begin{cases} x_{t+1} = \Gamma x_t \; + \; BEp_t \; + \; Cw_t \\ p_{t+1} = \qquad (\Gamma^*)^{-1}p_t - PCw_t \\ w_{t+1} = \qquad\qquad\qquad Dw_t \end{cases} \tag{3.5}$$

For an optimal process, p_t, given by (2.23), and x_t are bounded. Conversely, let x_t, p_t be a bounded solution of (3.5) and let $u_t = Kx_t + Ep_t$. Using the first equation in (3.5) and (2.7) we obtain

$$x_{t+1} = Ax_t + Bu_t + Cw_t, \quad u_t = Kx_t + r_t.$$

Therefore, by Theorem 2.3, x_t, u_t is an optimal process.

Now, consider the linear $(2n+m)$-dimensional system (3.5). Since Γ is stable and therefore $(\Gamma^*)^{-1}$ is antistable, the state space is decomposed as the direct sum of three invariant subspaces, the n-dimensional stable subspace $\mathcal{M}_+$, the n-dimensional antistable subspace $\mathcal{M}_-$, and m-dimensional center manifold $\mathcal{M}_0$, being the subspaces spanned by the generalized eigenvectors of the coefficient matrix of (3.5) corresponding to the eigenvalues of modulus less than one, greater than one and one respectively. The evolution of the entire linear system (3.5) is a superposition of three motions, the one on $\mathcal{M}_+$ which tend asymptotically to zero as $t \to \infty$, the one on $\mathcal{M}_-$ which is unbounded, and the one on $\mathcal{M}_0$ which is harmonic.

We remark that almost all solutions of (3.5) are unbounded and hence do not correspond to optimal processes. By Theorem 2.3, p_t, as a unique bounded solution of (2.16), is given by (2.23). Therefore, in view of (3.1) and (3.2), p_t must be harmonic in the optimal solution, i.e.

$$p_t^0 = \sum_{j=1}^{m} p^{(j)} \beta_j e^{i\theta_j t}, \tag{3.6}$$

where

$$p^{(j)} = -\Gamma^*(e^{i\theta_j}\Gamma^* - I_n)^{-1}PCe_j \tag{3.7}$$

with e_j being the j:th columns of the identity matrix I_m. Consequently, r_t, given by (2.18), must also be harmonic and is given by

$$r_t = y_t^0 = \sum_{j=1}^{m} y^{(j)} \beta_j e^{i\theta_j t}, \tag{3.8}$$

where

$$y^{(j)} = (B^*PB + R)^{-1}B^*(e^{i\theta_j}\Gamma^* - I)^{-1}PCe_j, \tag{3.9}$$

and therefore there is a matrix

$$Y := (y^{(1)}, y^{(2)}, \dots, y^{(m)}), \tag{3.10}$$

which does not depend on the unknown $\beta_1, \beta_2, \ldots, \beta_m$, such that

$$r_t = y_t^0 = Yw_t. \tag{3.11}$$

This implies that (3.5) may be replaced with

$$\begin{cases} x_{t+1} = \Gamma x_t + (BY + C)w_t \\ w_{t+1} = Dw_t \end{cases}, \tag{3.12}$$

the orbits of which are bounded and fill the $(n+m)$-dimensional subspace $\mathcal{M}_+ \oplus \mathcal{M}_0$. Note that equations (3.6)–(3.12) have been derived without resorting to the condition $\det \Gamma \neq 0$, so this condition is no longer needed. We have established that any solution of (3.12) together with

$$u_t = Kx_t + Yw_t \tag{3.13}$$

yields an optimal process. The equations (3.12) coincide with the system equations (2.11) with w_t given by (3.1) if we use the control (3.13). So the regulator (3.13) gives us an optimal process. But we can not use this regulator since the process w_t is not available through observations.

Next, consider two ideas of identification of the unknown w_t. We will see that both of them will fail. In the first we consider w_t as part of the state and try to construct an observer to estimate it from x_t. Indeed, the standard reduced-order observer is obtained by setting

$$z_t := w_t - Lx_t \tag{3.14}$$

so that

$$z_{t+1} = (D - LC)z_t + (DL - LCL - LA)x_t - LBu_t.$$

Then the observer will have the same structure, namely

$$\hat{z}_{t+1} = (D - LC)\hat{z}_t + (DL - LCL - LA)x_t - LBu_t$$

but with an initial condition which is an arbitrary estimate of $z_0 = w_0 - Lx_0$, say $\hat{z}_0 = 0$. Since $\tilde{z}_t := z_t - \hat{z}_t$ satisfies

$$\tilde{z}_{t+1} = (D - LC)\tilde{z}_t \tag{3.15}$$

and (C, D) is an observable pair (provided C is full rank), the *pole placement theorem* implies that L can be chosen so as to give $D - LC$ any desired spectrum; in particular we can make it stable. Then, by Theorem 2.3, the control law (3.13) could be replaced by

$$u_t = (K + YL)x_t + Y\hat{z}_t,$$

since $\epsilon_t := \tilde{z}_t \to 0$ as $t \to \infty$. Unfortunately, however, the corresponding closed-loop system will not be strictly stable since, as a simple calculation reveals, its characteristic polynomial will contain the characteristic polynomial χ_D of D as a factor. In fact, this will also be the case for a regulator based on a full-order observer. This is of course a manifestation of the fact that (3.12) is not a stabilizable system.

A second unworkable idea is based on the observation that the unknown amplitudes $\beta_1, \beta_2, \ldots, \beta_m$ can be determined exactly in a finite number of steps by choosing L in (3.15) so that all eigenvalues of $D - LC$ are all zero and hence $\tilde{z}_t \to 0$ in at most m steps so that $z_t, t = m, m + 1, \ldots,$ can be determined exactly. Then, by (3.14), x_t is completely known, and hence so is also y_t^0 as given by Theorem 2.3 and by (3.8). It may therefore seem reasonable to try to use a control law

$$u_t = Kx_t + y_t^0, \tag{3.16}$$

where y_t^0 is known and precomputed instead of being obtained via feedback. However, such a regulator will not be robust with respect to errors in the frequencies $\theta_1, \theta_2, \dots, \theta_m$.

To see this, let us first remark that, if f_t and g_t are harmonic sequences, i.e.

$$f_t = \sum_{j=1}^{m} f^{(j)} e^{i\theta_j t} \quad \text{and} \quad g_t = \sum_{j=1}^{m} g^{(j)} e^{i\theta_j t}, \tag{3.17}$$

and W is an arbitrary matrix of appropriate dimensions, then

$$\mathrm{M}\{f_t^* W g_t\} = \sum_{j=1}^{m} \sum_{k=1}^{m} f^{(j)*} W g^{(k)} \lim_{T\to\infty} \{\frac{1}{T} \sum_{t=0}^{T} e^{i(\theta_k - \theta_j)t}\}. \tag{3.18}$$

The limit in this expression does exist, and it is one if $\theta_j = \theta_k$ and zero otherwise. Therefore, since the frequencies $\theta_1, \theta_2, \dots, \theta_m$ are distinct,

$$\mathrm{M}\{f_t^* W g_t\} = \sum_{j=1}^{m} f^{(j)*} W g^{(j)}. \tag{3.19}$$

(If the frequencies were not distinct, the expression becomes somewhat more complicated but the idea would be the same.)

Now, returning to the question of robustness, let us suppose that the frequencies $\hat\theta_1, \hat\theta_2, \dots, \hat\theta_m$ used in computing the control law (3.16) are not the same as those driving the system, being estimates of $\theta_1, \theta_2, \dots, \theta_m$. Then the control really becomes

$$\hat u_t = Kx_t + \hat y_t^0, \tag{3.20}$$

where $\hat y_t^0$ is (3.8) computed with respect to the estimated frequencies $\hat\theta_1, \hat\theta_2, \dots, \hat\theta_m$. Clearly $\hat y_t^0 \to y_t^0$ and thus $\hat u_t \to u_t$ as $\hat\theta_j \to \theta_j$, $j = 1, \dots, m$, for any fixed t. Nevertheless, as we shall see, the regulator will not be robust. To see this, note that, by (2.27) in Theorem 2.3, the cost of using the control (3.20) is

$$\hat\Phi = \Phi_{\min} + \mathrm{M}\{(\hat u_t - Kx_t - y_t^0)^* \hat R (\hat u_t - Kx_t - y_t^0)\},$$

i.e. the increase in the cost function is

$$\Delta\Phi = \mathrm{M}\{(\hat y_t{}^0 - y_t^0)^* \hat R (\hat y_t^0 - y_t^0)\}.$$

Now, assuming that all $\hat\theta_j$ are different, we have

$$\hat y_t^0 - y_t^0 = \sum_{\hat\theta_j \neq \theta_j} (\hat y^{(j)} e^{i\hat\theta_j t} - y^{(j)} e^{i\theta_j t})$$

and, therefore, in view of (3.19),

$$\Delta\Phi = \sum_{\hat\theta_j \neq \theta_j} (\hat y^{(j)*} \hat R \hat y^{(j)} + y^{(j)*} \hat R y^{(j)}).$$

Consequently, as $\hat{\theta}_j \to \theta_j$ for $j = 1, 2, \ldots, m$, and hence

$$\hat{y}^{(j)} = \hat{R}^{-1}B^*(I - e^{i\hat{\theta}_j}\Gamma^*)^{-1}Ce_j \to y^{(j)} = \hat{R}^{-1}B^*(I - e^{i\theta_j}\Gamma^*)^{-1}Ce_j$$

we have

$$\Delta\Phi \to 2 \sum_{\hat{\theta}_j \neq \theta_j} y^{(j)*}\hat{R}y^{(j)} =: \Delta\Phi_0 \tag{3.21}$$

so that an arbitrary small mistake in the estimation of frequencies $\theta_1, \theta_2, \ldots, \theta_m$ produces a jump $\Delta\Phi_0$ in the cost function. Due to this discontinuity the control law (3.16) is not optimal for practical purposes.

Let us now return to (3.12), the state space of which has the decomposition $\mathcal{M}_+ \oplus \mathcal{M}_0$ so that all orbits converge to the center manifold $\mathcal{M}_0$ of harmonic solutions. Since the component in $\mathcal{M}_+$ – let us call it z_t^+ – tends asymptotically to zero,

$$\mathrm{M}\{|z_t^+|^2\} = 0,$$

and therefore only the harmonic component $\begin{bmatrix} x_t^0 \\ w_t^0 \end{bmatrix}$ in $\mathcal{M}_0$ contributes to the cost function. Consequently, in Theorem 2.3, u_t has the form

$$u_t = Kx_t^0 + y_t^0 + \epsilon_t \tag{3.22}$$

where still ϵ_t is any sequence satisfying (2.25). Here x_t^0 and y_t^0 are the harmonic solutions

$$x_t^0 = \sum_{j=1}^{m} x^{(j)}\beta_j e^{i\theta_j t} \quad \text{and} \quad y_t^0 = \sum_{j=1}^{m} y^{(j)}\beta_j e^{i\theta_j t}, \tag{3.23}$$

where

$$\begin{cases} x^{(j)} = (e^{i\theta_j}I - \Gamma)^{-1}(By^{(j)} + Ce_j) \\ y^{(j)} = (B^*PB + R)^{-1}B^*(e^{i\theta_j}\Gamma^* - I)^{-1}PCe_j \end{cases} \tag{3.24}$$

The expression for y_t^0 has already been derived above (see (3.8), (3.9)) and the one for x_t^0 is then obtained from (3.12).

Next, consider the regulator (1.9), which we shall write in a slightly different form. In fact, let us introduce the new control

$$y_t = u_t - Kx_t, \tag{3.25}$$

in terms of which (1.9) may be written

$$D(\sigma)y_t = N(\sigma)x_t, \tag{3.26}$$

where $N(\lambda)$ is the matrix polynomial

$$N(\lambda) = M(\lambda) - D(\lambda)K. \tag{3.27}$$

Moreover, the system (1.1) becomes

$$x_{t+1} = \Gamma x_t + By_t + Cw_t, \tag{3.28}$$

where $\Gamma := A - BK$ is stable, as pointed out in Section 2. We shall say that the regulator (1.9), or, equivalently, the regulator defined by (3.26) and (3.25), is optimal if the solution (x_t, u_t) of the closed-loop system (1.1), (3.26) is an optimal process in our problem to minimize (1.4) subject to (2.11), (3.1) and (3.2), for any initial conditions.

Theorem 3.2. *Let* (x_t^0, y_t^0) *be the harmonic optimal process defined by* (3.23), (3.24), *and let* w_t *be given by* (3.2). *Then the regulator* (1.9) *is optimal for the problem to control the system* (1.1) *so as to minimize* (1.4) *if the closed-loop system* (3.26), (3.28) *is asymptotically stable and has a harmonic solution* (x_t^0, y_t^0) *which coincides with* (3.23).

Proof. Because of stability any solution of (3.26), (3.28) has the property that $y_t = y_t^0 + \epsilon_t$, where $\epsilon_t \to 0$ exponentially as $t \to \infty$. Therefore, recalling that $y_t^0 = r_t$, (3.25) implies that $u_t = Kx_t + r_t + \epsilon_t$ where $\mathrm{M}\{|\epsilon_t|^2\} = 0$. Also, x_t, which tends exponentially to x_t^0, satisfies the admissibility condition (1.7). Consequently, by Theorem 2.3, the process x_t, u_t is optimal for the problem to control the system (1.1) so as to minimize (1.4), i.e. the regulator (1.9) is optimal for the problem to control the system (1.1). □

We are now in a position to formulate the general principles that need to be followed in designing an optimal, robust and universal regulator for the control problem in Section 1. The goal is to construct an optimal regulator (3.26) in which the matrix polynomials $D(\lambda)$ and $N(\lambda)$ are chosen so that

(i) the closed-loop system (3.26), (3.28) is asymptotically stable,
(ii) the closed-loop system (3.26), (3.28) has the same harmonic solutions x^0, y^0 as (3.23), (3.24) for any complex amplitudes $\beta_1, \beta_2, \dots, \beta_m$,
(iii) the matrix polynomials $D(\lambda)$ and $N(\lambda)$ in the regulator (3.26) do not depend on $\beta_1, \beta_2, \dots, \beta_m$,
(iv) the regulator (3.26) is robust in the sense that if it is determined from estimates $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_m$ of the frequencies $\theta_1, \theta_2, \dots, \theta_m$, then the value $\Phi(\hat{\theta}, \theta)$ of the cost functional must be continuous in $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_m$ so that, in particular, $\Phi(\hat{\theta}, \theta) \to \Phi(\theta, \theta)$ as $\hat{\theta} \to \theta$. (This is a somewhat nonstandard use of the concept "robust".)

By Theorem 3.2, conditions (i) and (ii) imply that the regulator (3.26) will be optimal. It will be shown in Section 5 that (iv) is a consequence of (i) and (ii).

4. The general representation of a stabilizing regulator

We have now reduced the problem of Section 1 to that of designing a regulator (3.26), independent of $\beta_1, \beta_2, \dots, \beta_m$, rendering the closed-loop system (3.26), (3.28) asymptotically stable and having a harmonic solution (x_t^0, y_t^0) coinciding with the harmonic solution of system (1.1) obtained by applying the nonrobust and unrealizable regulator of Section 2. This section will be devoted to the stability condition (i).

More precisely, in this section we shall parameterize the class

$$D(\sigma)y_t = N(\sigma)x_t \tag{4.1}$$

of stabilizing linear regulators, where σ is the forward shift operator $\sigma x_t = x_{t+1}$ and $D(\lambda), N(\lambda)$ are real $k \times k$ and $k \times n$ matrix polynomials such that the leading coefficient of $D(\lambda)$ is nonsingular and $\deg N \le \deg D$ so that $D(\lambda)^{-1}N(\lambda)$ is a proper rational matrix function. Consequently,

$$\begin{cases} x_{t+1} = \Gamma x_t + By_t + v_t \\ D(\sigma)y_t = N(\sigma)x_t \end{cases} \tag{4.2}$$

is the closed-loop system under consideration. We recall that Γ is a stable matrix and that $x_t \in \mathbb{C}^n, y_t \in \mathbb{C}^k$.

Therefore, from now on, we shall take (4.2) with Γ stable as the starting point of the analysis of this section. Let $\Psi_x(\lambda)$ and $\Psi_y(\lambda)$ be the transfer functions from the input v_t to the outputs x_t and y_t respectively of this new system. They are defined by

$$\begin{cases} (\lambda I_n - \Gamma)\Psi_x(\lambda) = B\Psi_y(\lambda) + I_n \\ N(\lambda)\Psi_x(\lambda) = D(\lambda)\Psi_y(\lambda) \end{cases}, \tag{4.3}$$

and consequently

$$\Psi_x(\lambda) = (\lambda I_n - \Gamma)^{-1}(B\Psi_y(\lambda) + I_n). \tag{4.4}$$

Condition (i) at the end of Section 3 is precisely the condition that (4.2) is stable. To say that (4.2) is stable is to say that

$$\Xi(\lambda) = \begin{bmatrix} (\lambda I_n - \Gamma) & -B \\ N(\lambda) & -D(\lambda) \end{bmatrix} \tag{4.5}$$

is a stable matrix polynomial, i.e. that $\det \Xi(\lambda) \neq 0$ for $|\lambda| \geq 1$. Now recall the following definition.

Definition 4.1. The regulator (4.1) is said to be *stabilizing* for the system

$$x_{t+1} = \Gamma x_t + By_t + v_t \tag{4.6}$$

if the closed-loop system (4.2) is stable and $\Psi_x(\infty) = \Psi_y(\infty) = 0$.

The last requirement insures causality in the sense that x_t, y_t in (4.2) will depend on $v_j, j < t$, only. We also introduce

Definition 4.2. The regulators $D_1(\sigma)y_t = N_1(\sigma)x_t$ and $D_2(\sigma)y_t = N_2(\sigma)x_t$ of the type (4.1) are called *equivalent* if there exist matrix polynomials D_0, N_0 such that

$$D_1 = M_1D_0, \quad N_1 = M_1N_0, \quad D_2 = M_2D_0, \quad N_2 = M_2N_0$$

for some stable matrix polynomials M_1, M_2.

It is clear that systems (4.2) with equivalent regulators have the same transfer functions Ψ_x, Ψ_y. Moreover, if one regulator is stabilizing, then so is the other. The following lemma, which is also of independent interest, completely characterizes those regulators (4.1) which satisfy condition (i).

Lemma 4.3. *Let* Γ *in* (4.2) *be a stable matrix. Let* $\rho(\lambda)$ *be an arbitrary real scalar stable polynomial, and let* $R(\lambda)$ *be an arbitrary real* $k \times n$ *matrix polynomial such that* $\deg R(\lambda) < \deg \rho(\lambda)$. *Then the regulator* (4.1) *with*

$$\begin{cases} D(\lambda) = R(\lambda)B + \rho(\lambda)I_k \\ N(\lambda) = R(\lambda)(\lambda I_n - \Gamma) \end{cases} \tag{4.7}$$

is stabilizing for the system (4.6), *and, for this regulator,*

$$\Psi_y(\lambda) = \frac{R(\lambda)}{\rho(\lambda)}, \quad \det \Xi(\lambda) = (-1)^k \rho(\lambda)^k \det(\lambda I - \Gamma), \tag{4.8}$$

where Ξ *is given by* (4.5). *The class of regulators* (4.1), (4.7) *contains all stabilizing regulators in the sense that any other stabilizing regulator is equivalent to one in this class.*

We note that, since the coefficients in (1.1) are real, so are the polynomials ρ and R. For the complex case we would need the polynomials to be complex. Lemma 4.3 may be deduced from the Youla parameterization, but it is simpler to give an independent proof.

Proof. Set $\Gamma_\lambda := \lambda I - \Gamma$, $\delta(\lambda) := \det \Gamma_\lambda$ and $Q(\lambda) := \delta(\lambda)\Gamma_\lambda^{-1}$. From (4.5) and (4.7) we obtain

$$\det \Xi = \delta \det[N\Gamma_\lambda^{-1}B - D] = \delta \det(-\rho I) = (-1)^k \delta \rho^k,$$

which is a stable polynomial. Therefore (4.2) is stable. Now, in view of (4.4) and (4.7),

$$N\Psi_x = N\Gamma_\lambda^{-1}(B\Psi_y + I) = RB\Psi_y + R$$

and

$$D\Psi_y = RB\Psi_y + \rho\Psi_y,$$

and so $N\Psi_x = D\Psi_y$ yields

$$R(\lambda) = \rho(\lambda)\Psi_y(\lambda).$$

Thus we have established (4.8). Since $\deg R < \deg \rho$, $\Psi_y(\infty) = 0$, and consequently, by (4.4), $\Psi_x(\infty) = 0$. Therefore the regulator (4.1) is stabilizing.

Now, let $D'(\sigma)y_t = N'(\sigma)x_t$ be an arbitrary stabilizing regulator, and let Ψ'_x, Ψ'_y be the transfer functions formed in analogy with Ψ_x, Ψ_y. Then we have $\Psi'_x(\infty) = 0$ and $\Psi'_y(\infty) = 0$, and $\det \Xi'$ is stable. Here

$$\begin{aligned} \det \Xi' = \det \begin{bmatrix} \Gamma_\lambda & -B \\ N' & -D' \end{bmatrix} &= \det \Gamma_\lambda \det(N'\Gamma_\lambda^{-1}B - D') \\ &= \delta \det(S\delta^{-1}) = \delta^{-(k-1)} \det S, \end{aligned} \tag{4.9}$$

where

$$S = N'QB - \delta D' \tag{4.10}$$

is a matrix polynomial. From (4.9) we have $\det S = \delta^{k-1} \det \Xi'$, so S must be stable. Let

$$S_c = S^{-1} \det S \tag{4.11}$$

be the adjoint matrix polynomial. In accordance with (4.3) we have

$$\begin{cases} \Gamma_\lambda \Psi'_x(\lambda) = B\Psi'_y(\lambda) + I \\ N'(\lambda)\Psi'_x(\lambda) = D'(\lambda)\Psi'_y(\lambda) \end{cases}.$$

Consequently

$$D'\Psi'_y = N'\Gamma_\lambda^{-1}(B\Psi'_y + I) = \delta^{-1}N'Q(B\Psi'_y + I),$$

i.e. $S\Psi'_y = -N'Q$, and hence, in view of (4.11),

$$\Psi'_y = -\frac{S_c N'Q}{\det S}. \tag{4.12}$$

Now, let us take

$$R(\lambda) = -S_c(\lambda)N'(\lambda)Q(\lambda), \quad \rho(\lambda) = \det S(\lambda), \tag{4.13}$$

and let D and N be defined correspondingly by (4.7). Since $\Psi'_y(\infty) = 0$, we have $\deg R < \deg \rho$. Moreover ρ is stable. Therefore, as proved above, $Dy_t = Nx_t$ is a stabilizing regulator, and $\Psi_y = R/\rho$, so we must have $\Psi_y = \Psi'_y$, and consequently, by (4.4), $\Psi_x = \Psi'_x$. Since $\deg R < \deg \rho$, we have $\det D(\lambda) \not\equiv 0$, and (4.4) implies $\det \Psi_x \not\equiv 0$. Consequently the second of relations (4.3) can be written $D^{-1}N = \Psi_y\Psi_x^{-1}$. But we also have $D'^{-1}N' = \Psi_y\Psi_x^{-1}$ and so

$$N' = D'\Psi_y\Psi_x^{-1} = D'D^{-1}N. \tag{4.14}$$

Let the $k \times k$ matrix polynomial M be the greatest left common divisor of D and N, i.e.

$$D = MD_0, \qquad N = MN_0, \tag{4.15}$$

where D_0 and N_0 are left coprime matrix polynomials. Since $Dy_t = Nx_t$ is a stabilizing regulator, M is stable, and, since $\det D \not\equiv 0$, $\det M \not\equiv 0$ and $\det D_0 \not\equiv 0$. From (4.14) we have $N' = D'D_0^{-1}N_0$, so setting $M' := D'D_0^{-1}$, we obtain

$$D' = M'D_0, \qquad N' = M'N_0. \tag{4.16}$$

Since D_0 and N_0 are left coprime, there exist matrix polynomials Π_1 and Π_2 such that

$$D_0\Pi_1 + N_0\Pi_2 = I.$$

(See, e.g., [10].) Therefore

$$M' = M'(D_0\Pi_1 + N_0\Pi_2) = D'\Pi_1 + N'\Pi_2$$

is a matrix polynomial. Since $D'y_t = N'y_t$ is a stabilizing regulator, M' is stable. From (4.15)–(4.16) we now see that the regulators $D'y_t = N'x_t$ and $Dy_t = Nx_t$ are equivalent. $\square$

This lemma provides us with a complete answer to the question of how to satisfy condition (i) at the end of Section 3: We can use a regulator of the type (4.1) with D, N defined by (4.7) for some ρ and R, and, modulo equivalence, the regulators of this type are all the stabilizing regulators. From (4.7) we see that the leading coefficient of $D(\lambda)$ is nonsingular and that $\deg N \le \deg D$, which implies that (4.1) is a causal regulator, i.e. $D(\lambda)^{-1}N(\lambda)$ is proper.

That the limit in the cost function (1.4) does exist for any stabilizing regulator in the class presented in this section, as required by Proposition 3.1, is a consequence of the following theorem which is also of interest in its own right and will be needed in Sections 5 and 7.

Theorem 4.4. *Let $u_t = Kx_t + y_t$. Then the limit*

$$\Phi = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \Lambda(x_t, u_t) \tag{4.17}$$

exists for any process (x_t, u_t) defined by a stabilizing regulator (4.1), *and it takes the form*

$$\Phi = \beta^* \Omega \beta, \tag{4.18}$$

where the $m \times m$ matrix Ω depends continuously on $\theta_1, \theta_2, \dots, \theta_m$, A, B, C, Q, R and S and the parameters of the regulator polynomials $D(\lambda)$ and $N(\lambda)$. Moreover, the admissibility condition (1.7) *is satisfied.*

Proof. Since the closed-loop system

$$\begin{cases} x_{t+1} = \Gamma x_t + By_t + Cw_t \\ D(\sigma)y_t = N(\sigma)x_t \end{cases}$$

is stable, x_t and y_t tend asymptotically to the harmonic solutions

$$\tilde{x}_t = \sum_{j=1}^{m} \tilde{x}^{(j)} \beta_j e^{i\theta_j t}, \quad \tilde{y}_t = \sum_{j=1}^{m} \tilde{y}^{(j)} \beta_j e^{i\theta_j t}, \tag{4.19}$$

where

$$\begin{bmatrix} \tilde{x}^{(j)} \\ \tilde{y}^{(j)} \end{bmatrix} = \begin{bmatrix} (e^{i\theta_j} I - \Gamma) & -B \\ N(e^{i\theta_j}) & -D(e^{i\theta_j}) \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} C\beta_j e_j, \tag{4.20}$$

as $t \to \infty$, and, *a fortiori*, x_t satisfies the admissibility condition (1.7). Now,

$$\Phi = \mathrm{M}\{ \begin{bmatrix} x_t \\ y_t \end{bmatrix}^* W \begin{bmatrix} x_t \\ y_t \end{bmatrix} \},$$

where the symmetric matrix

$$W = \begin{bmatrix} Q + SK + K^*S^* + K^*RK & S + K^*R \\ S^* + RK & R \end{bmatrix}$$

depends continuously on Q, R, S and K. In view of the fact that $x_t - \tilde{x}_t \to 0$ and $y_t - \tilde{y}_t \to 0$, this can be written

$$\Phi = \mathrm{M}\{ \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix}^* W \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix} \}. \tag{4.21}$$

Consequently, (4.18) follows from (4.19), (4.20) and (3.19), and the analysis leading to (3.19) shows that the limit in (4.17) exists, as claimed. Moreover, the stabilizing solution P to the algebraic Riccati equation (2.6) depends continuously on A, B, Q, S and R, and hence so does K as defined by (2.5). Consequently, in view of (4.20), the statement on continuity holds. □

5. The existence and design of the realizable, robust, optimal regulator

We now turn to the other requirements for the regulator which are enumerated at the end of Section 3.

Condition (ii), which we consider first, implies that $\Psi_x(\lambda)$ and $\Psi_y(\lambda)$ satisfy certain interpolation relations insuring that (4.2) has the harmonic solution (3.23)–(3.24). As seen from (3.1) and (3.2), the harmonic solution of (4.2) with $v_t = Cw_t$ is

$$x_t = \sum_{j=1}^{m} \tilde{x}^{(j)} \beta_j e^{i\theta_j t}, \quad y_t = \sum_{j=1}^{m} \tilde{y}^{(j)} \beta_j e^{i\theta_j t}, \tag{5.1}$$

where

$$\tilde{x}^{(j)} = \Psi_x(e^{i\theta_j})Ce_j, \quad \tilde{y}^{(j)} = \Psi_y(e^{i\theta_j})Ce_j. \tag{5.2}$$

Here e_j is the j:th unit axis vector of dimension m, i.e. the j:th column of the identity matrix I_m.

The conditions $x_t = x_t^0$ and $y_t = y_t^0$ required for optimality hold for all $\beta_1, \beta_2, \dots, \beta_m$ if and only if the following interpolation conditions are valid:

$$\Psi_x(e^{i\theta_j})Ce_j = x^{(j)}, \quad \Psi_y(e^{i\theta_j})Ce_j = y^{(j)}, \quad j = 1, 2, \dots, m. \tag{5.3}$$

The relations for Ψ_x in (5.3) follow from the ones for Ψ_y. In fact, using (4.4) and the expression for $x^{(j)}$ in (3.23), we transform the interpolation relation (5.3) for $x^{(j)}$ into

$$(B\Psi_y(e^{i\theta_j}) + I)Ce_j = By^{(j)} + Ce_j, \tag{5.4}$$

which follows from the second set of equations (5.3). Therefore Ψ_x may be omitted from the subsequent analysis. By replacing $y^{(j)}$ in (5.3) by the expression in (3.24), the remaining interpolation conditions become

$$\Psi_y(e^{i\theta_j})Ce_j = (B^*PB + R)^{-1}B^*(e^{i\theta_j}\Gamma^* - I)^{-1}PCe_j \quad j = 1, 2, \dots, m \tag{5.5}$$

and are thus independent of $\beta_1, \beta_2, \dots, \beta_m$.

Now, inserting $\Psi_y(\lambda) = \frac{R(\lambda)}{\rho(\lambda)}$, as prescribed by Lemma 4.3, into the interpolation conditions (5.5), we obtain

$$R(e^{i\theta_j})Ce_j = \rho(e^{i\theta_j})\Theta(e^{i\theta_j})PCe_j, \quad j = 1, 2, \dots, m, \tag{5.6}$$

where

$$\Theta(\lambda) = (B^*PB + R)^{-1}B^*(\lambda\Gamma^* - I)^{-1}, \tag{5.7}$$

so R and ρ must be chosen to satisfy (5.6) and the conditions of Lemma 4.3. If $c_j := Ce_j \neq 0$, the j:th interpolation condition (5.6) can be written

$$R(e^{i\theta_j}) = \rho(e^{i\theta_j})\Theta(e^{i\theta_j})Pc_j(c_j^*c_j)^{-1}c_j^* + \tilde{R}_j, \tag{5.8}$$

where $\tilde{R}_j$ is an arbitrary matrix such that $\tilde{R}_j c_j = 0$. It is clear that there exists a solution $R(\lambda)$ of (5.6) for each ρ of sufficiently high degree.

Obviously the interpolation relations (5.6) do not contain the unknown complex amplitudes $\beta_1, \beta_2, \dots, \beta_m$. Therefore R and ρ and, consequently D and N in (4.7), will not depend on $\beta_1, \beta_2, \dots, \beta_m$ either, and hence condition (iii) is satisfied. Recall that an optimal regulator with this property is called *universal*.

To prove that condition (iv) holds, note that, by Theorem 4.4, the cost function Φ depends continuously on the parameters of the regulator polynomials $D(\lambda)$ and $N(\lambda)$, which in turn depends continuously on the polynomials $R(\lambda)$ and $\rho(\lambda)$ via (4.7). The regulator is determined by fixing a $\rho(\lambda)$ of sufficiently high degree and determining $R(\lambda)$ from (5.8) with $\theta_1, \theta_2, \dots, \theta_m$ exchanged for $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_m$. It remains to prove that this $R(\lambda)$ depends continuously on $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_m$. To this end, observe that (5.8) is a (generally underdetermined) system of linear equations in the coefficients of $R(\lambda)$, and therefore the question is reduced to deciding that the coefficient matrix of this linear system has full rank, which is the case since it is a (block) Vandermonde matrix corresponding to distinct points on the unit circle.

Thus we have established a general formula for the required universal optimal regulator. In fact, Lemma 4.3 gives us the complete class of stabilizing regulators (satisfying condition (i)), and the interpolation conditions (5.6) are equivalent to condition (ii).

We summarize our results in the following theorem. Before that, however, let us recall the problem formulation for the complex problem: Given the system (2.11) with the external disturbance given by (3.1) and (3.2), find a realizable regulator (1.6) satisfying condition (1.7) such that the regulator does not depend on the unknown complex amplitudes $\beta_1, \beta_2, \dots, \beta_m$ and the cost functional (1.4) is minimized for all $\beta_1, \beta_2, \dots, \beta_m$.

Theorem 5.1. *Let $\rho(\lambda)$ be an arbitrary real scalar monic stable polynomial, and let $R(\lambda)$ be a real matrix polynomial satisfying the interpolation conditions* (5.6) *and having degree less than that of $\rho(\lambda)$. Moreover, let $D(\lambda)$ and $N(\lambda)$ be given by* (4.7) *respectively. Then the regulator*

$$D(\sigma)u_t = [N(\sigma) + D(\sigma)K]x_t, \tag{5.9}$$

is optimal for the problem posed in Section 3 *and it renders the complete closed-loop system asymptotically stable, and therefore* (1.7) *holds. It is also robust in the sense of condition (iv) at the end of Section* 3*, and it is universal in the sense that it does not depend on the unknown complex amplitudes $\beta_1, \beta_2, \dots, \beta_m$. Finally, modulo equivalence, all universal optimal regulators are formed in this way.*

Now, in view of Proposition 3.1 and Theorem 4.4, this regulator is also an optimal robust and universal regulator for the original real problem posed in Section 1. Also, modulo equivalence, all universal optimal regulators are formed in this way.

By taking (F, G, H, L) to be a (minimal or nonminimal) realization of $D(\lambda)^{-1}N(\lambda)$, i.e., a representation

$$H(\lambda I - F)^{-1}G + L = D(\lambda)^{-1}N(\lambda), \tag{5.10}$$

we can write the regulator (5.9) in the form

$$\begin{cases} z_{t+1} = Fz_t + Gx_t \\ u_t = Hz_t + Jx_t \end{cases} \tag{5.11}$$

with $J = K + L$, K being the gain (2.5). We observe, however, that $D(\lambda)$ and $N(\lambda)$ need not be coprime and that any left common factors are canceled in determining a minimal realization $(F, G, H, J - K)$ of $D^{-1}N$. Therefore, for the sake of robustness, a nonminimal realization may be preferable.

Remark 5.2. Theorem 5.1 states that the regulator (5.11) is optimal in any wider class of regulators which is in harmony with condition (1.7). In particular, no nonlinear or nonrealizable regulator will yield a smaller value of the cost functional (1.4). The same, of course, holds for the real problem of Section 1. (To see that there is a linear optimal regulator, use the formulation in the footnote on page 9 to avoid the question of existence of limits.)

Remark 5.3. Note that, since R and ρ are real, the interpolation condition $R(e^{i\theta_j})c_j = \rho(e^{i\theta_j})\Theta(e^{i\theta_j})Pc_j$ is equivalent to $R(e^{-i\theta_j})c_j = \rho(e^{-i\theta_j})\Theta(e^{-i\theta_j})Pc_j$.

Let us next consider the question of determining $\rho(\lambda)$ and $R(\lambda)$. Clearly, there is a considerable degree of design freedom here. If [2]

$$\det C^*C \neq 0 \tag{5.12}$$

and consequently $m \leq n$, we can always choose $\rho(\lambda)$ to be of degree two and take $R(\lambda)$ of the form

$$R(\lambda) = R_0\lambda + R_1, \tag{5.13}$$

where R_0, R_1 are real matrices. To prove this, insert (5.13) into the interpolation conditions (5.6), yielding the system of equations

$$\begin{cases} a_j \cos\theta_j + b_j & = \operatorname{Re}\{\rho(e^{i\theta_j})\Theta(e^{i\theta_j})PCe_j\} \\ a_j \sin\theta_j & = \operatorname{Im}\{\rho(e^{i\theta_j})\Theta(e^{i\theta_j})PCe_j\} \end{cases} \tag{5.14}$$

in the real k-vectors $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m$, defined by

$$a_j := R_0Ce_j, \quad b_j := R_1Ce_j, \quad j = 1, 2, \dots, m, \tag{5.15}$$

where, as before, e_j is the j:th column vector in the identity matrix I_m. The solution of (5.14) is unique provided no θ_j is a multiple of π. Otherwise, the second equation is trivial so the first alone determines the (nonunique) solution. Given $a_j, b_j, j = 1, 2, \dots, m$, the matrices R_0 and R_1 can be obtained from (5.15). To this end, form the $k \times m$ matrix polynomial

$$S(\lambda) := [a_1, a_2, \dots, a_m]\lambda + [b_1, b_2, \dots, b_m] \tag{5.16}$$

of degree one. Then, if condition (5.12) holds, $R(\lambda)$ can be solved from

$$R(\lambda)C = S(\lambda). \tag{5.17}$$

In fact, if $\det C^*C \neq 0$,

$$R(\lambda) = S(\lambda)(C^*C)^{-1}C^* \tag{5.18}$$

is a (in general nonunique) solution of (5.17). On the other hand, if $\det C^*C = 0$, the degrees of $\rho(\lambda)$ and $R(\lambda)$ may need to be increased.

In the case $m \geq n$, the degree of ρ will in general increase with m.

[2]The observation in Remark 5.3 may allow us to remove some redundant columns in C.

6. A simple numerical example

Consider the problem to design a universal optimal regulator for the scalar plant

$$y_{t+1} + ay_{t-1} = u_t + f_t \tag{6.1}$$

with the external disturbance

$$f_t = \alpha_0 + \alpha_1 \cos(\omega_1 t + \varphi_1) + \alpha_1 \cos(\omega_2 t + \varphi_2), \tag{6.2}$$

where, as before, the frequencies ω_1 and ω_2 are known, while the amplitudes α_0, α_1, α_2 and the phases φ_1, φ_2 are unknown. Hence this disturbance includes a bias as well as harmonic oscillations. The problem is to find an admissible regulator (1.8) which is stabilizing in the sense that $t^{-1}y_t \to 0$ as $t \to \infty$ and universal in the sense that it does not depend on α_0, α_1, α_2 and φ_1, φ_2, and which minimizes the cost functional

$$\Phi = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} (y_t^2 + u_t^2) \tag{6.3}$$

for any values of $\alpha_0, \alpha_1, \alpha_2, \varphi_1$ and φ_2. Introducing the state

$$x_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}, \tag{6.4}$$

the plant equations (6.1) can be written in the state form

$$x_{t+1} = Ax_t + Bu_t + Cw_t, \tag{6.5}$$

where

$$A = \begin{bmatrix} 0 & -a \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{6.6}$$

and

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad w_t = \begin{bmatrix} \frac{1}{2}\alpha_2 e^{-i\varphi_2} e^{-i\omega_2 t} \\ \frac{1}{2}\alpha_1 e^{-i\varphi_1} e^{-i\omega_1 t} \\ \alpha_0 \\ \frac{1}{2}\alpha_1 e^{i\varphi_1} e^{i\omega_1 t} \\ \frac{1}{2}\alpha_2 e^{i\varphi_2} e^{i\omega_2 t} \end{bmatrix} = \begin{bmatrix} \beta_{-2} e^{i\theta_{-2} t} \\ \beta_{-1} e^{i\theta_{-1} t} \\ \beta_0 e^{i\theta_0 t} \\ \beta_1 e^{i\theta_1 t} \\ \beta_2 e^{i\theta_2 t} \end{bmatrix} \tag{6.7}$$

if we reformulate the problem according to the footnote on page 9, or

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad w_t = \begin{bmatrix} \alpha_0 \\ \alpha_1 e^{i\varphi_1} e^{i\omega_1 t} \\ \alpha_2 e^{i\varphi_2} e^{i\omega_2 t} \end{bmatrix} = \begin{bmatrix} \beta_0 e^{i\theta_0 t} \\ \beta_1 e^{i\theta_1 t} \\ \beta_2 e^{i\theta_2 t} \end{bmatrix} \tag{6.8}$$

if, as we shall do here, we inbed our problem in the complex optimization problem as described in Proposition 3.1. Moreover,

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R = 1 \tag{6.9}$$

are the parameters in the cost function (1.4). It is easy to see that the corresponding algebraic Riccati equation (2.6) has the unique stabilizing solution

$$P = \begin{bmatrix} p_{11} & 0 \\ 0 & p_{11} - 1 \end{bmatrix}, \tag{6.10}$$

where

$$p_{11} = \frac{a^2}{2} + \sqrt{\frac{a^4}{4} + 1}, \tag{6.11}$$

and the gain (2.5) is given by

$$K = \begin{bmatrix} 0 & \kappa \end{bmatrix}, \qquad \kappa = \frac{ap_{11}}{1 + p_{11}} \tag{6.12}$$

In fact,

$$B^*PB + R = 1 + p_{11} > 0,$$

and the corresponding feedback matrix (2.7),

$$\Gamma = \begin{bmatrix} 0 & -\gamma \\ 1 & 0 \end{bmatrix}, \qquad \gamma = \frac{a}{1 + p_{11}}. \tag{6.13}$$

has all its eigenvalues strictly inside the unit circle. If $a > 0$, there will be a pair $\pm i\sqrt{\gamma}$ of imaginary eigenvalues, and if $a < 0$ a pair $\pm\sqrt{|\gamma|}$ of real ones.

Let us now choose some (real) polynomial

$$\rho(\lambda) = \lambda^5 + \rho_1\lambda^4 + \rho_2\lambda^3 + +\rho_3\lambda^2 + \rho_4\lambda + \rho_5,$$

which in the present example must be of degree five, having all its roots strictly inside the unit circle. The parameters $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ will be available for tuning in order to improve the overall design. Next, we want to determine a real 1×2 matrix polynomial $R(\lambda)$ of degree at most four which satisfies the interpolation conditions (5.8) in which we choose $\tilde{R}_j = 0$ for each j. Since $c_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for each j and, as a simple calculation shows, (5.7) is given by

$$\Theta(\lambda) = \frac{1}{1 + p_{11} + a\lambda^2} \begin{bmatrix} -1 & -\lambda \end{bmatrix},$$

we need to find an $R(\lambda)$ which satisfies the interpolation conditions

$$R(e^{i\theta_j}) = \begin{bmatrix} r^{(j)} & 0 \end{bmatrix} \qquad r^{(j)} = \frac{-p_{11}\rho(e^{i\theta_j})}{1 + p_{11} + ae^{2i\theta_j}}. \tag{6.14}$$

Clearly such an $R(\lambda)$ must have the form

$$R(\lambda) = \begin{bmatrix} r(\lambda) & 0 \end{bmatrix} \qquad r(e^{i\theta_j}) = r^{(j)} \quad \text{for all } j. \tag{6.15}$$

As explained in Remark 5.3, we only need to satisfy this interpolation condition for $j = 0, 1, 2$; then the condition is automatically satisfied for $j = -1, -2$. We can therefore use the format expressed by (6.8). Then, except for $j = 0$ which yield a real condition, we obtain a real equation for both the real and the imaginary part. Consequently, the coefficients of the real scalar polynomial

$$r(\lambda) = r_1\lambda^4 + r_2\lambda^3 + r_3\lambda^2 + r_4\lambda + r_5 \tag{6.16}$$

must be the solution of the system of linear equations

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \cos 4\theta_1 & \cos 3\theta_1 & \cos 2\theta_1 & \cos \theta_1 & 1 \\ \sin 4\theta_1 & \sin 3\theta_1 & \sin 2\theta_1 & \sin \theta_1 & 0 \\ \cos 4\theta_2 & \cos 3\theta_2 & \cos 2\theta_2 & \cos \theta_2 & 1 \\ \sin 4\theta_2 & \sin 3\theta_2 & \sin 2\theta_2 & \sin \theta_2 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} = \begin{bmatrix} r^{(0)} \\ \Re r^{(1)} \\ \Im r^{(1)} \\ \Re r^{(2)} \\ \Im r^{(2)} \end{bmatrix} \tag{6.17}$$

where $\Re$ denotes real part and $\Im$ imaginary part.

We are now in a position to describe a class of optimal universal regulators. In fact, from (4.7), we obtain

$$\begin{cases} D(\lambda) = r(\lambda) + \rho(\lambda) \\ N(\lambda) = \begin{bmatrix} \lambda r(\lambda) & \gamma r(\lambda) \end{bmatrix}, \end{cases} \tag{6.18}$$

so, in view of Theorem 5.1, (6.4) and (6.12), the optimal regulator corresponding to ρ is given by

$$[\rho(\sigma) + r(\sigma)]\, u_t = \sigma r(\sigma) y_t + [\kappa\rho(\sigma) + a r(\sigma)]\, y_{t-1}, \tag{6.19}$$

i.e., the transfer function of the regulator from the output to the control is

$$F(\lambda) = \frac{\lambda^2 r(\lambda) + \kappa\rho(\lambda) + a r(\lambda)}{\lambda[\rho(\lambda) + r(\lambda)]}. \tag{6.20}$$

We stress again that we have one such universal optimal regulator for each admissible choice of ρ.

As an alternative to solving (6.17), we may use Lagrange's interpolation formula to obtain

$$r(\lambda) = \sum_{j=-2}^{2} r^{(j)} \pi_j(\lambda), \tag{6.21}$$

where

$$\pi_j(\lambda) = \prod_{k \neq j} \frac{\lambda - \lambda_k}{\lambda_j - \lambda_k}; \quad \lambda_k = e^{i\theta_k}. \tag{6.22}$$

Here we must use the symmetric formulation (6.7) to obtain a real polynomial $r(\lambda)$.

In Figure 6.1 we show a simulation for the situation that $a = -0.8$, $\theta_1 = 0.3$ and $\theta_2 = 1.0$. To illustrate the amount of damping, we depict the output y_t both for the case that there is no control ($u_t = 0$) and for the optimal universal regulator corresponding to the polynomial ρ with roots $-0.3, -0.4 \pm 0.2, -0.5 \pm 0.3$.

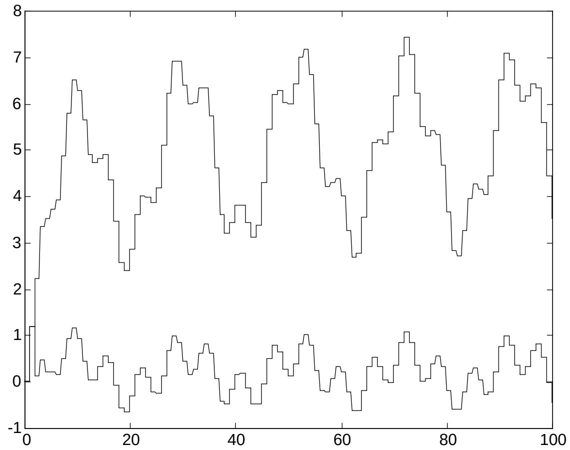

Figure 6.1

The choice of ρ must be made with some care, since it may drastically affect the transient. In fact, the transient behavior does not affect the value of the cost function.

Next let us consider what happens if the true frequencies of the system are not quite the ones used in computing the regulator but there is an estimation error. Figure 6.2 shows the outputs obtained if the regulator of Figure 6.1, based on the freqences $\theta_1 = 0.3$ and $\theta_2 = 1.0$, is applied to a system with true frequences $\theta_1 = 0.5$ and $\theta_2 = 1.5$.

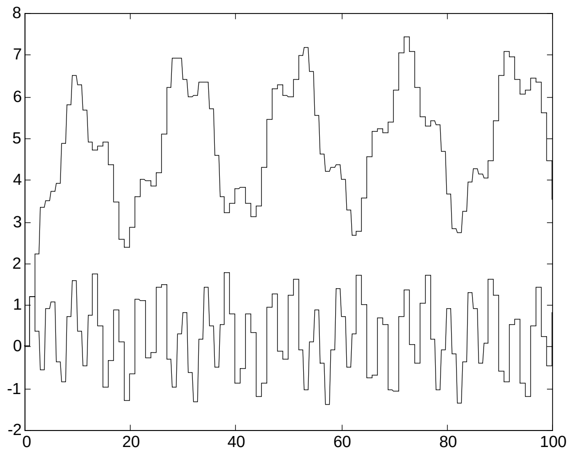

Figure 6.2

As we can see the regulator still behaves reasonably despite the large errors in the frequency estimates.

7. The stochastic case

A natural question to ask is whether the regulator of Theorem 5.1 remains optimal if the amplitudes $\beta_1, \beta_2, \ldots, \beta_m$ are allowed to be random variables or processes and the cost functional (1.4) to be minimized is replaced by

$$\Phi = \lim_{T\to\infty} \mathrm{E}\{\frac{1}{T}\sum_{t=0}^{T} \Lambda(x_t, u_t)\}, \tag{7.1}$$

where $\mathrm{E}\{\cdot\}$ denotes mathematical expectation. As before, we assume that $\{x_k; k \leq t\}$ is known at time t, so the regulator should be chosen in some suitable class of feedback laws

$$u_t = f(t, x_0, x_1, \dots, x_t) \tag{7.2}$$

with the property that a condition similar to (1.7) holds.

In the case that $\beta_1, \beta_2, \dots, \beta_m$ are random variables, it is not hard to convince oneself that the answer to this question is affirmative. In fact, in the deterministic case studied above, the same optimal regulator can be used for each fixed set of values of $\beta_1, \beta_2, \dots, \beta_m$. Therefore, summing over a probability measure will yield the same optimal regulator as in the deterministic case.

As it turns out, and this is the topic of this section, more general external disturbances w_t may be considered. In fact, we may consider a control system

$$x_{t+1} = Ax_t + Bu_t + Cw_t \tag{7.3}$$

with w_t being the solution of a "harmonic" linear stochastic system

$$w_{t+1} = Dw_t + \xi_{t+1}, \quad w_0 = \beta, \tag{7.4}$$

where D and β are given by (3.4), β is a random vector with mean $\bar{\beta} := \mathrm{E}\{\beta\}$, $\{\xi_0, \xi_1, \xi_2, \dots\}$ is a zero-mean, white-noise process with $\xi_0 := \beta - \bar{\beta}$, i.e.

$$\mathrm{E}\{\xi_s\xi_t^*\} = \rho_t\delta_{st}, \quad \mathrm{E}\{\xi_t\} = 0, \tag{7.5}$$

and $\{|\rho_t|\}_{t=0}^{\infty}$ is an ℓ_1 sequence, i.e.

$$\sum_{t=0}^{\infty} |\rho_t| < \infty. \tag{7.6}$$

The noise model (7.4) does not damp past white noise exponentially, as does the usual "colored noise" model for which D has all its eigenvalues strictly inside the unit circle. Consequently, (7.6) is needed to decrease the influence of past white noise as time goes on and is actually the natural condition insuring that the process $\{w_t\}$ has bounded covariance. In fact,[3],

$$w_t = \sum_{j=1}^{m} e_j\bar{\beta}_j e^{i\theta_j t} + \sum_{j=1}^{m} e_j \sum_{k=0}^{t} (\xi_k)_j e^{i\theta_j(t-k)}, \tag{7.7}$$

where the condition (7.6) insures that

$$\mathrm{E}\{w_t w_t^*\} = D^t\bar{\beta}\bar{\beta}^*(D^*)^t + \sum_{k=0}^{t} D^{t-k}\rho_k(D^*)^{t-k}$$

[3]We recall that e_j is the j:th column vector in I_m.

is bounded for all $t \in \mathbb{Z}_+$. This should be compared with the deterministic case considered before, which is obtained by setting $\rho_t \equiv 0$. As we have full state information, it is no restriction to assume that x_0 is deterministic.

We restrict our attention to the following class of admissible control laws. Let $\mathcal{L}$ be the class of linear feedback laws (7.2) corresponding to regulators

$$D(\sigma)u_t = M(\sigma)x_t, \tag{7.8}$$

as defined in Section 1, such that the closed-loop system consisting of (7.3) and (7.8) is asymptotically stable.

We remark that adding a white noise term, which is independent of other system noise, to the left member of (7.3) does not alter the problem. In fact, for any $f \in \mathcal{L}$, the contribution of this white noise to the processes x_t and u_t produces an additive contribution to $\Lambda(x_t, u_t)$ which tends to zero as $t \to \infty$ and hence does not affect the cost Φ.

Theorem 7.1. *Consider the control system* (7.3) *with the external disturbance* w_t *being defined by* (7.4)*, or, equivalently, by* (7.7)*, where* ξ_t *satisfies* (7.5) *and* (7.6)*. Then the limit in* (7.1) *exists for all* $f \in \mathcal{L}$*. Moreover, if* $\hat{f}$ *corresponds to an optimal regulator of Theorem* 5.1*,* $\hat{f} \in \mathcal{L}$*, and* $\hat{f}$ *is also optimal, with respect to the cost functional* (7.1)*, for the problem to control* (7.3) *in the class* $\mathcal{L}$*.*

Proof. The white-noise process ξ_t can be represented in the form

$$\xi_t = L_t \eta_t,$$

where L_t is a matrix-valued function and η_t is a zero-mean, p-dimensional, normalized white noise, i.e.

$$\mathrm{E}\{\eta_s \eta_t^*\} = I\delta_{st}, \quad \mathrm{E}\{\eta_t\} = 0. \tag{7.9}$$

Then

$$w_t = \bar{w}_t + \sum_{\ell=1}^{p} \sum_{k=0}^{t} w_t(k,\ell)(\eta_k)_\ell, \tag{7.10}$$

where

$$w_t(k,\ell) = \sum_{j=1}^{m} e_j \gamma_{jk\ell} e^{i\theta_j t}, \quad \gamma_{jk\ell} = (L_k)_{j\ell} e^{-i\theta_j k} \tag{7.11}$$

and

$$\bar{w}_t = \sum_{j=1}^{m} e_j \bar{\beta} e^{i\theta_j t}. \tag{7.12}$$

Clearly, an admissible process x_t, u_t defined via a control law (7.8) with $f \in \mathcal{L}$ has a representation of similar form, namely

$$x_t = \bar{x}_t + \sum_{\ell=1}^{p} \sum_{k=0}^{t-1} x_t(k,\ell)(\eta_k)_\ell \tag{7.13}$$

$$u_t = \bar{u}_t + \sum_{\ell=1}^{p} \sum_{k=0}^{t-1} u_t(k,\ell)(\eta_k)_\ell, \tag{7.14}$$

where $\{\bar{x}_t\}, \{\bar{u}_t\}, \{x_t(k,\ell)\}$ and $\{u_t(k,\ell)\}$ are deterministic vector sequences. More precisely, since $\bar{x}_t = \mathrm{E}\{x_t\}$,

$$\bar{x}_{t+1} = A\bar{x}_t + B\bar{u}_t + C\bar{w}_t, \quad \bar{x}_0 = x_0, \tag{7.15}$$

and since $x_t(k,\ell) = \mathrm{E}\{x_t(\eta_k)_\ell\}$ for $t \geq k+1$,

$$x_{t+1}(k,\ell) = Ax_t(k,\ell) + Bu_t(k,\ell) + Cw_t(k,\ell), \quad x_{k+1}(k,\ell) = Cw_k(k,\ell) \tag{7.16}$$

for $t = k+1, k+2, \dots$.

In view of (7.11) and (7.12), these equations all have the same structure, namely that of the deterministic case, and they differ only in the amplitudes of the harmonic external disturbances, the quantities which do not affect the optimal regulator in the deterministic case. Also, it is easy to check that

$$\begin{aligned}\mathrm{E}\{\Lambda(x_t,u_t)\} &= \Lambda(\bar{x}_t,\bar{u}_t) + \sum_{\ell=1}^{p}\sum_{k=0}^{t-1} \Lambda(x_t(k,\ell),u_t(k,\ell)) \\ &= \Lambda(\bar{x}_t,\bar{u}_t) + \sum_{\ell=1}^{p}\sum_{k=0}^{\infty} \Lambda(x_t(k,\ell),u_t(k,\ell)),\end{aligned} \tag{7.17}$$

if we agree to define $x_t(k,\ell)$ and $u_t(k,\ell)$ to be zero for $k \geq t$. Consequently,

$$\mathrm{E}\{\frac{1}{T}\sum_{t=0}^{T}\Lambda(x_t,u_t)\} = \frac{1}{T}\sum_{t=0}^{T}\Lambda(\bar{x}_t,\bar{u}_t) + \sum_{\ell=1}^{p}\sum_{k=0}^{\infty}\left[\frac{1}{T}\sum_{t=k+1}^{T}\Lambda(x_t(k,\ell),u_t(k,\ell)\right]. \tag{7.18}$$

We would like to be able to take the limit in this expression so that

$$\Phi = \bar{\Phi} + \sum_{\ell=1}^{p}\sum_{k=0}^{\infty}\Phi_{k\ell}, \tag{7.19}$$

where

$$\bar{\Phi} = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\Lambda(\bar{x}_t,\bar{u}_t) \tag{7.20}$$

and

$$\Phi_{k\ell} = \lim_{T\to\infty}\frac{1}{T}\sum_{t=k+1}^{T}\Lambda(x_t(k,\ell),u_t(k,\ell)). \tag{7.21}$$

This, of course, needs to be justified. We proceed next to doing precisely this.

Let us first address the question of existence of the limits (7.20) and (7.21). Due to the linearity of the control laws in $\mathcal{L}$,

$$\bar{u}_t = f(t, \bar{x}_0, \bar{x}_1, \dots, \bar{x}_t),$$

and

$$u_t(k, \ell) = f(t, x_0(k, \ell), x_1(k, \ell), \dots, x_t(k, \ell)).$$

whenever the control law $f \in \mathcal{L}$ is applied to the stochastic problem. But then, by Theorem 4.4, the limits exist, and $\bar{x}_t$ and $x_t(k, \ell)$ satisfy the admissibility condition (1.7).

Secondly, Theorem 4.4 also implies that $\Phi_{k\ell} = \gamma_{k\ell}^* \Omega \gamma_{k\ell}$, where Ω varies with the choice of $f \in \mathcal{L}$ and $\gamma_{k\ell} = \operatorname{col}(\gamma_{1k\ell}, \gamma_{2k\ell}, \dots, \gamma_{mk\ell})$ is defined as in (7.11). Consequently, since $\gamma_{k\ell} = L_k e_\ell$ and $L_k L_k^* = \rho_k$, $|\Phi_{k\ell}| \leq \kappa_1 |\rho_k|$ for some constant $\kappa_1 > 0$, and therefore, in view of (7.6),

$$\sum_{k=0}^{\infty} \Phi_{k\ell} < \infty.$$

It remains to prove that

$$\lim_{T \to \infty} \sum_{k=0}^{\infty} \left[\frac{1}{T} \sum_{t=k+1}^{T} \Lambda(x_t(k, \ell), u_t(k, \ell)) \right] = \sum_{k=0}^{\infty} \Phi_{k\ell}. \tag{7.22}$$

But, in view of (3.18), there is a uniform bound

$$\left| \frac{1}{T} \sum_{t=k+1}^{T} \Lambda(x_t(k, \ell), u_t(k, \ell)) \right| \leq \kappa_2 |\rho_k|$$

for some $\kappa_2 > 0$, and consequently (7.22) follows by a dominated convergence argument.

Consequently, we have now decomposed the problem into a countable number of separate, uncoupled deterministic problems of the same structure as that of Theorem 5.1, namely Problem $\bar{P}$ to minimize (7.20) given the system (7.15) and Problems $P_{k\ell}$ to minimize (7.21) given (7.16). These problems differ only in the values of the initial state and the amplitudes (and phases) of the external disturbance. But, by Theorem 5.1, the optimal regulator does not depend on these quantities. Consequently, it solves all these problems simultaneously, and therefore it yields the minimum of the the functional (7.19).

Now, let $\hat{f}$ be the linear control law corresponding to an optimal regulator (5.11) in the deterministic problem of Theorem 5.1. Clearly, $\hat{f} \in \mathcal{L}$. Moreover, Problem $\bar{P}$ has the optimal solution

$$\bar{u}_t = \hat{f}(t, \bar{x}_0, \bar{x}_1, \dots, \bar{x}_t), \tag{7.23}$$

and, for each (k, ℓ), Problem $P_{k\ell}$ has the solution

$$u_t(k, \ell) = \hat{f}(t, x_0(k, \ell), x_1(k, \ell), \dots, x_t(k, \ell)). \tag{7.24}$$

In Problem $P_{k\ell}$, $\gamma_{1k\ell}, \gamma_{2k\ell}, \dots, \gamma_{mk\ell}$ play the role of the unknown amplitudes. Now, inserting (7.23) and (7.24) into (7.14) and applying (7.13) yields

$$u_t = \hat{f}(t, x_0, x_1, \dots, x_t), \tag{7.25}$$

by linearity. Consequently, (7.25) is optimal for the stochastic problem as claimed. □

We note that the decomposition (7.15), (7.16) and (7.19) is analogous to the one used in [20], so a natural question is whether the admissible class of regulators could be extended to include nonlinear control laws as in [20]. However, this leads to technical difficulties related to the existence of the limits (7.20) and (7.21) and the validity of (7.22).

8. Conclusions

In this paper we present a complete characterization of all regulators which (i) stabilize a linear system with additive harmonic disturbances with known frequencies but unknown amplitudes and phases, (ii) minimize an infinite-horizon quadratic cost function and (iii) are universal in the sense that the regulators do not depend on the unknown amplitudes and phases and are optimal for all choices of these. These *optimal universal regulators* are linear, but we show that they are optimal in a wide class of nonlinear regulators. Finally, we show that these regulators are also optimal universal regulators (in a natural sense) for a corresponding stochastic problem.

We stress that our solutions are optimal in the sense stated in this paper only, and that other desirable design specifications may not be satisfied for an arbitrary universal optimal regulator. Therefore it is an important property of our procedure that it allows for a considerable degree of design freedom. How this design freedom is to be used may be the topic of a future paper.

As pointed out to us by one of the referees, related optimal control problems have been studied in some recent papers [13, 28], but with different problem formulations.

Appendix A. Necessity of the frequency-domain condition.

Let us prove the assertion on page 4: *if the frequency-domain condition* (1.13) *fails in a strong way, then there exists an external disturbance* w_t *such that* $\inf \Phi = -\infty$. Since the frequency-domain condition (1.13) is invariant under feedback and the pair (A, B) is stabilizable, we will assume without loss of generality that A is stable.

Suppose that the frequency-domain condition fails strongly for the values x^0, u^0, λ^0 so that

$$\Lambda(x^0, u^0) < 0 \tag{A.1}$$

for

$$\lambda^0 x^0 = Ax^0 + Bu^0, \qquad \lambda^0 = e^{i\theta}, \quad \theta \in \mathbb{R}. \tag{A.2}$$

We need to find a sequence $x_t^{(j)}, u_t^{(j)}$ of admissible processes (which is defined via a sequence of regulators (1.9)) such that $\Phi^{(j)} \to -\infty$, where $\Phi^{(j)}$ are the corresponding values of the functional (1.4).

Let us first consider the case when the admissible process is allowed to be complex. Consider a process and perturbation of the type

$$x_t = x^0 e^{i\theta t} + \Delta x_t, \quad u_t = u^0 e^{i\theta t}, \quad w_t = w^0 e^{i\theta t}, \tag{A.3}$$

where $w^0 \in \mathbb{C}^m$ and Δx_t remain to be defined. Admissibility requires that the (1.1) and (1.9) must hold, where (1.9) is a stabilizing regulator. By Lemma 4.3, we may take

$$D(\lambda) = R(\lambda)B + \rho(\lambda)I_k, \quad M(\lambda) = R(\lambda)(\lambda I_n - A) \tag{A.4}$$

for some polynomials $\rho(\lambda)$ and $R(\lambda)$ with the properties prescribed by Lemma 4.3. Using (1.1) we transform (1.9) into

$$\rho(\sigma)u_t = R(\sigma)Cw_t, \tag{A.5}$$

so the system (1.1), (1.9) is equivalent to (1.1), (A.5). Since $C \neq 0$, we can choose w^0 so that $c^0 := Cw^0 \neq 0$. Equation (A.5) is satisfied if

$$\rho(\lambda^0)u^0 = R(\lambda^0)c^0. \tag{A.6}$$

For example, we can take as $R(\lambda)$ the constant matrix

$$R(\lambda) \equiv \rho(\lambda^0)u^0[(c^0)^*c^0]^{-1}(c^0)^*. \tag{A.7}$$

In accordance with Lemma 4.3 we here take $\rho(\lambda)$ to be any stable scalar polynomial of degree ≥ 1. Then (1.9) is a stabilizing regulator. In view of (A.2) and (A.3), we see that (1.1) is satisfied if

$$\Delta x_t = \Delta x^0 e^{i\theta t}, \quad \Delta x^0 = (\lambda^0 I - A)^{-1}c^0, \tag{A.8}$$

and therefore equations (A.3) and (A.8) with $c^0 = Cw^0 \neq 0$ define an admissible process. Note that x^0, u^0 in (A.1) and (A.2) may be replaced by $\lambda x^0, \lambda u^0$ for an arbitrary $\lambda \in \mathbb{C}$. Next, we construct a sequence of admissible processes by replacing x^0, u^0 in (A.3) by $x^{(j)} = \lambda_j x^0, u^{(j)} = \lambda_j u^0$ where $|\lambda_j| \to \infty$, yielding the admissible processes

$$x_t^{(j)} = \lambda_j x^0 e^{i\theta t} + \Delta x_t, \quad u_t^{(j)} = \lambda_j u^0 e^{i\theta t}, \quad j = 1, 2, 3, \dots. \tag{A.9}$$

By formula (A.7) we have

$$R_j(\lambda) \equiv \lambda_j \rho(\lambda^0)u^0[(c^0)^*c^0]^{-1}(c^0)^*, \tag{A.10}$$

where we have taken a $\rho(\lambda)$ which does not depend on j. (Therefore, to our sequence of admissible processes (A.9), there corresponds a sequence of stabilizing regulators (1.9) with D and M defined by (A.4) and $R(\lambda) \equiv R_j$.)

Consider now the corresponding sequence of cost functionals Φ. By (1.4) and (1.5), we have

$$\Phi^{(j)} = \begin{pmatrix} \lambda_j x^0 + \Delta x^0 \\ \lambda_j u^0 \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix} \begin{pmatrix} \lambda_j x^0 + \Delta x^0 \\ \lambda_j u^0 \end{pmatrix} = |\lambda_j|^2 \Lambda(x^0, u^0) + O(|\lambda_j|). \tag{A.11}$$

Therefore $\Phi^{(j)} \to -\infty$ as $|\lambda_j| \to \infty$. This concludes the proof of the complex case.

Next, consider the real case. We now have the system (1.1) with w_t defined by (1.2), or, more generally, the system consisting of (2.11), (3.1) and (3.2) with $v_t = Cw_t$ real. The matrices A, B, C are real, as are the coefficients of the form $\Lambda(x, u)$ and of the polynomials D and M in (1.9). The admissible process must also be real, so the process (A.3) can no longer be used. Let us therefore rename C and w_t in (1.1), (1.2) as C^0 and w_t^0, as in the footnote on page 9. Then (1.1) becomes

$$x_{t+1} = Ax_t + Bu_t + C^0 w_t^0, \quad x_0 = a \tag{A.12}$$

with $w_t^0 = \operatorname{Re} w_t$, $w_t = \operatorname{col}(\beta_1 e^{i\omega_1 t}, \dots, \beta_\nu e^{i\omega_\nu t})$ and $\beta_j = \alpha_j e^{i\varphi_j}$. Certainly the system (A.12) has the previously considered form (2.11) provided we put $v_t = Cw_t = C^0 w_t^0$, $C = [C^0, C^0]$, $w_t = \frac{1}{2}\begin{bmatrix} w_t^{(1)} \\ \bar{w}_t^{(1)} \end{bmatrix}$, $m = 2\nu$ and $\theta_j = \omega_j$, $\theta_{j+\nu} = \omega_{j+\nu}$ for $j = 1, 2, \dots, \nu$. We have shown that the complex processes

$$\tilde{x}_t = (x^0 + \Delta x)e^{i\theta t}, \quad \tilde{u}_t = u^0 e^{i\theta t} \tag{A.13}$$

defined by (A.3) and (A.8) with $c^0 = C^0 w^0$ satisfy the complex system

$$\tilde{x}_{t+1} = A\tilde{x}_t + B\tilde{u}_t + C^0 \tilde{w}_t^0, \quad \tilde{w}_t = w^0 e^{i\theta t}. \tag{A.14}$$

Therefore the real process

$$x_t = \operatorname{Re} \tilde{x}_t, \quad u_t = \operatorname{Re} \tilde{u}_t \tag{A.15}$$

satisfies (A.12). Now consider the stabilizing regulator (1.9) for the plant (A.14) which as we have seen may be rewritten as (A.5), which in our present notation reads

$$\rho(\sigma)\tilde{u}_t = R(\sigma)C^0 \tilde{w}_t^0. \tag{A.16}$$

Choosing real polynomials ρ and R here, the process (A.15) satisfies the equation

$$\rho(\sigma)u_t = R(\sigma)C^0 w_t^0. \tag{A.17}$$

implying that (A.15) will be the admissible process. Therefore we have to find a real stable scalar polynomial $\rho(\lambda)$ and a real matrix polynomial $R(\lambda)$, $\deg R < \deg \rho$, satisfying (A.16), which is equivalent to

$$\rho(\lambda^0)u^0 = R(\lambda^0)C^0 w^0. \tag{A.18}$$

Without loss of generality let us assume that $\sin\theta \neq 0$. (If needed, we can perturb the value of θ a little in (A.1) and (A.2).) Let w^0 be real, $c^0 = C^0 w^0 \neq 0$ and $R(\lambda) = R_0 + R_1\lambda$ with real matrix coefficients R_0, R_1. Moreover, let $\rho(\lambda)$ be any stable polynomial, $\deg\rho > 2$, and set $\rho(\lambda^0)u^0 = u' + iu''$, where u', u'' are real. Equation (A.18) gives

$$u' = (R_0 + R_1 \cos\theta)c^0, \quad u'' = (R_1 \sin\theta)c^0.$$

To satisfy these equations we can take, for example,

$$R_1 = \frac{1}{\sin\theta} u''[(c^0)^* c^0]^{-1}(c^0)^*, \quad R_0 = (u' - R_1 c^0 \cos\theta)[(c^0)^* c^0]^{-1}(c^0)^*.$$

Then the process (A.15) is admissible. Next, we find the value of the functional (1.4) for this process. To this end, set $z = \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{C}^{n+k}$, and let $z_t = \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \hat{z}e^{i\theta t}$ be an admissible process, where $\hat{z} = \begin{bmatrix} x^0 + \Delta x \\ u^0 \end{bmatrix} = a + ib$, a, b real. Then, for any real Hermitian form $\Lambda(z)$, we have

$$\Lambda(z) = \Lambda(\operatorname{Re} z) + \Lambda(\operatorname{Im} z), \tag{A.19}$$

and $\operatorname{Re} z_t = a\cos(\theta t) - b\sin(\theta t)$. A simple calculation yields

$$\frac{1}{T}\sum_{t=1}^{T} \Lambda(\operatorname{Re} z_t) = \frac{1}{2}[\Lambda(a) + \Lambda(b)] + O(\frac{1}{T}),$$

and therefore, using (A.18), we have

$$\Phi = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \Lambda(\operatorname{Re} z_t) = \frac{1}{2}[\Lambda(a) + \Lambda(b)] = \frac{1}{2}\Lambda(\hat{z}).$$

Consider again the sequence of admissible processes $x_t{}^{(j)}, u_t{}^{(j)}$ with $\hat{z} = \begin{bmatrix} \lambda_j x^0 + \Delta x \\ \lambda_j u^0 \end{bmatrix}$, where $\lambda_j \to \infty$. The corresponding values of Φ are $\Phi_j = \frac{1}{2}\lambda_j^2 \Lambda(x^0, u^0) + O(\lambda_j)$. Since $\Lambda(x^0, u^0) < 0$, we obtain $\Phi_j \to -\infty$ as $\lambda_j \to \infty$, and hence $\inf \Phi = -\infty$, as claimed.

Appendix B. Admissibility of the control (2.24).

Lemma B.1. *Consider the solution x_t of*

$$x_{t+1} = \Gamma x_t + f_t \tag{B.1}$$

with $x_0 = 0$, where the matrix Γ is stable in the sense that there is a $K > 0$ and a $\gamma > 0$ such that $|\Gamma^t| \le K e^{-\gamma t}$ for $t \ge 0$. Then, for any $t \ge 0$,

$$|x_t|^2 \le \frac{K^2}{1 - e^{-\gamma}} \sum_{s=0}^{t-1} e^{-\gamma(t-s-1)} |f_s|^2 \tag{B.2}$$

Proof. Since $x_t = \sum_{s=0}^{t-1} \Gamma^{t-s-1} f_s$, we have $|x_t| \le K \sum_{s=0}^{t-1} e^{-\gamma(t-s-1)} |f_s|$, and, consequently,

$$\begin{aligned} |x_t|^2 &\le \sum_{s_1=0}^{t-1}\sum_{s_2=0}^{t-1} K^2 e^{-\gamma(2t-s_1-s_2-2)} |f_{s_1}||f_{s_2}| \\ &\le \frac{1}{2}\sum_{s_1=0}^{t-1}\sum_{s_2=0}^{t-1} K^2 e^{-\gamma(2t-s_1-s_2-2)} (|f_{s_1}|^2 + |f_{s_2}|^2). \end{aligned}$$

Therefore, since $\sum_{s=0}^{t-1} e^{-\gamma(t-s-1)} \le \frac{1}{1-e^{-\gamma}}$, we obtain (B.2). □

Corollary B.2. *Under the conditions of Lemma* B.1 *we have*

$$|x_t|^2 \le \frac{K^2}{1 - e^{-\gamma}} \sum_{s=0}^{t-1} |f_s|^2. \tag{B.3}$$

Now, in (2.29), $f_t = B(r_t + \epsilon_t) + v_t$, and hence

$$|f_t|^2 \leq 4|B|(|r_t|^2 + |\epsilon_t|^2) + 2|v_t|^2.$$

But r_t and v_t are bounded, and consequently, by Lemma B.1, there are constants K_1 and K_2 such that $|x_t|^2 \leq K_1 + K_2 \sum_{s=0}^{t-1} |\epsilon_s|^2$ when $x_0 = 0$. Therefore $\mathrm{M}\{|\epsilon_s|^2\} = 0$ implies that $\frac{1}{t}|x_t|^2 \to 0$ as $t \to \infty$. Since Γ is stable, this is obviously true for any initial condition x_0.

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