UNIVERSAL REGULATORS FOR OPTIMAL TRACKING IN DISCRETE-TIME SYSTEMS AFFECTED BY HARMONIC DISTURBANCES*

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ABSTRACT. We consider the problem of controlling a discrete-time linear system by output feedback so as to have a second output z_t track an observed reference signal r_t. First, as a preliminary, we consider the problem of asymptotic tracking. i.e., to design a regulator such that $|z_t - r_t| \to 0$. This problem has been studied intensely in the literature, mainly in the continuous-time case. It is known that only under very special conditions does there exist a linear regulator which achieves this design goal and which is universal in the sense that it works for all reference signals and does not depend on them. On the other hand, if r_t is a harmonic signal with known frequencies but with unknown amplitudes and phases, there exist such regulators under mild conditions, provided the dimension of r_t is no larger than the number of controls. This is true even if the plant itself is corrupted by an unobserved additive harmonic disturbance w_t of the same type as r_t, if the dimension of w_t is no larger than the number of outputs available for feedback control.

However, if the first dimensionality condition is not satisfied, asymptotic tracking is not possible, but a steady state tracking error remains. Therefore we turn to another approach to the tracking problem, which also allows for damping of other system and control variables, and this is our main result. The measure of performance is given by a natural quadratic cost function. The object is to design an optimal regulator which is universal in the sense that it does not depend on the unknown amplitudes and phases of r_t and w_t and is optimal for all choices of r_t and w_t. We prove that an optimal universal regulator exists in a wide class of stabilizing and possibly nonlinear regulators under natural technical conditions and that this regulator is in fact linear, provided that the second dimensionality condition above is satisfied. On the other hand, if it is not satisfied, the existence of an optimal universal regulator is not a generic property, so as a rule no optimal universal regulator exists.

We provide complete solutions of all the problems described above.

1. Introduction

Consider a discrete-time linear control system

$$x_{t+1} = Ax_t + Bu_t + Ew_t \tag{1.1a}$$
$$y_t = Cx_t \tag{1.1b}$$
$$z_t = Hx_t + Ju_t \tag{1.1c}$$

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with a state $x_t \in \mathbb{R}^n$, two vector outputs $y_t \in \mathbb{R}^m$ and $z_t \in \mathbb{R}^\mu$, and two vector inputs, namely a control $u_t \in \mathbb{R}^k$ and an unobserved disturbance $w_t \in \mathbb{R}^\ell$ which we shall take to be harmonic with known frequencies but unknown amplitudes and phases. More precisely,

$$w_t = \sum_{j=1}^{N} w^{(j)} e^{i\theta_j t} \tag{1.2}$$

where the frequencies

$$-\pi < \theta_1 < \theta_2 < \cdots < \theta_N \leq \pi \tag{1.3}$$

are known, but the complex vector amplitudes $w^{(1)}, w^{(2)}, \dots, w^{(N)}$, in which the phases have been absorbed, are either completely unknown or zero. Consequently, some frequencies (1.3) may not be represented in w_t and have been included for notational purposes to be explained shortly.

In this paper we consider the problem to control the system (1.1) by feedback from the output y_t so as to have the output z_t track an observed μ-dimensional real reference signal

$$r_t = \sum_{j=1}^{N} r^{(j)} e^{i\theta_j t}, \tag{1.4}$$

which is harmonic with the known frequencies (1.3) but with complex vector amplitudes $r^{(1)}, r^{(2)}, \dots, r^{(N)}$ which are either completely unknown or zero so that certain frequencies (1.3) may not occur in r_t. The feedback configuration of this problem is described in the following flow diagram.

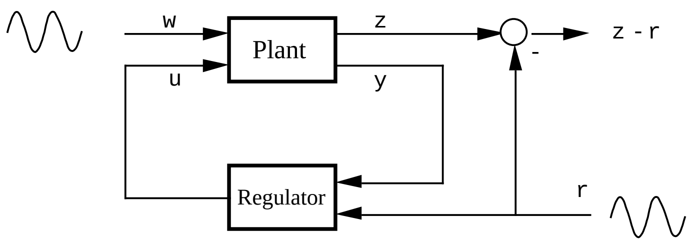

Figure 1.1: Feedback configuration

Many important engineering problems could be modeled in this way. Some examples are connected to industrial machines and helicopters [9, 10, 11, 12, 2, 28, 27], control of aircraft in the presence of wind shear [19, 23, 31], and control of the roll motion of a ship [14].

For notational convenience we use a common set of frequencies (1.3) for w_t and r_t, forcing us to set certain complex vector amplitudes equal to zero. To formalize this we introduce the index sets $\mathcal{J}_w, \mathcal{J}_r \subset \{1, 2, \dots, N\}$ of j for which $w^{(j)}$ and $r^{(j)}$ respectively are nonzero and arbitrary. Then

$$w_t = \sum_{j \in \mathcal{J}_w} w^{(j)} e^{i\theta_j t} \quad \text{and} \quad r_t = \sum_{j \in \mathcal{J}_r} r^{(j)} e^{i\theta_j t}. \tag{1.5}$$

Without loss of generality we assume that

$$\mathcal{J}_w \cup \mathcal{J}_r = \{1, 2, \dots, N\}.$$

Accordingly, we define the class $\mathcal{W}$ of disturbances and the class $\mathcal{R}$ of reference signals consisting of all signals w_t and r_t respectively obtained by letting $\{w^{(j)}\}_{j\in\mathcal{J}_w}$ and $\{r^{(j)}\}_{j\in\mathcal{J}_r}$ vary arbitrarily subject to the constraint that the signals (1.5) are real.

We assume that A, B, C, E, H, J are constant real matrices of appropriate dimensions such that (A, B) is stabilizable and (C, A) is detectable. Without loss of generality we may also assume that

$$\operatorname{rank} C = m \quad \text{and} \quad \operatorname{rank} E = \ell. \tag{1.6}$$

In fact, if the first condition is not satisfied, some components of y_t could be eliminated. Moreover, if E has linearly dependent columns, these could be combined without restriction. Clearly, (1.6) implies that $m \leq n$ and $\ell \leq n$.

Now, a possible criterion of performance for the tracking problem described above is given by

$$\Phi_0 = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \{|z_t - r_t|^2\}, \tag{1.7}$$

but, to allow for damping of internal system variables and the energy of control, we shall also consider a more general criterion of the type

$$\Phi = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \{\Lambda_0(x_t, u_t) + |z_t - r_t|^2\}, \tag{1.8}$$

where $\Lambda_0(x, u)$ is a real quadratic form

$$\Lambda_0(x, u) = \begin{pmatrix} x \\ u \end{pmatrix}^* \begin{pmatrix} Q_0 & S_0 \\ S_0^* & R_0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{1.9}$$

with properties to be specified in Section 5. (To insure that the infimum of Φ is not $-\infty$, we must of course introduce some condition on the quadratic form (1.9).) We note that the second functional (1.8) becomes a measure not only of the tracking accuracy but also of the forced oscillations in the closed-loop system. For the classes of admissible regulators to be defined next, these cost functions do not depend on initial conditions.

The object is to find, for suitable $p, q \in \mathbb{Z}$, a regulator

$$u_t = \varphi_t(y_t, y_{t-1}, \dots, y_{t-q}, r_t, r_{t-1}, \dots, r_{t-p}), \tag{1.10}$$

which is

(i) *stabilizing* in the sense that any process (x_t, u_t) satisfying the closed-loop system equations (1.1), (1.10) also satisfies the weak stability condition

$$\frac{1}{\sqrt{t}}|x_t| \to 0 \quad \text{as } t \to \infty; \tag{1.11}$$

(ii) *optimal* in the sense that the cost function (1.8) is minimized; and

(iii) *universal* in the sense that it simultaneously solves the complete family of optimization problems corresponding to different values of the complex vector amplitudes $\{w^{(j)}\}_{j\in\mathcal{J}_w}$ and $\{r^{(j)}\}_{j\in\mathcal{J}_r}$ and thus does not depend on these amplitudes.

Such a regulator will be referred to as an *optimal universal regulator* (OUR), and the class of regulators (1.10) satisfying conditions (i) and (ii) will be denoted $\mathcal{N}$. The stability condition (1.11) may at first sight seem somewhat unnatural, but, as we shall see in Section 6, it is the natural mathematical condition defining the largest class $\mathcal{N}$ for which statements of necessity and sufficiency can be made.

Removing the last term of (1.8) related to tracking we obtain some special cases of this problem which were studied in [21] and in [22] for the cases of complete and incomplete state information respectively.

In this paper we show that, under suitable technical conditions and provided $\ell \leq m$, the problem stated above has a solution in $\mathcal{N}$, and this solution happens to be a linear stabilizing regulator of type

$$M(\sigma)u_t = N(\sigma)y_t + L(\sigma)r_t, \tag{1.12}$$

where σ is the backward shift $\sigma y_t = y_{t+1}$ and $M(\lambda)$, $N(\lambda)$ and $L(\lambda)$ are real matrix polynomials, of dimensions $k \times k$, $k \times m$ and $k \times \mu$ respectively, with the property that $\det M(\lambda) \not\equiv 0$ and $M^{-1}N$ and $M^{-1}L$ are proper rational functions so that the regulator is nonanticipatory in the sense that u_t does not depend on future values of y_t and r_t, in harmony with (1.10). We shall denote by $\mathcal{L}$ the subclass of such linear regulators. Existence of an OUR in the subclass $\mathcal{L}$ itself can be established under somewhat milder technical conditions. The dimensionality condition $\ell \leq m$ is important. As in [22], it can be shown that if it fails then the existence of an optimal universal regulator becomes a nongeneric property. It means that no optimal universal regulator exists from a practical point of view if $\ell > m$.

The cost function (1.7) would of course be minimized if we could control (1.1a) so that

$$|z_t - r_t| \to 0 \quad \text{as} \qquad t \to \infty. \tag{1.13}$$

In fact, it would be zero. Therefore, asymptotic tracking appears as a special case in our analysis. This problem has been studied intensely in the literature at least in the continuous-time case; see, e.g., [1, 4, 5, 6, 7, 8, 13, 16] and references therein. The connection to this earlier work, developed in continuous time, is made evident by noting that the disturbance and reference signals (1.5) can be modeled as the output of a critically stable system

$$\begin{aligned} s_{t+1} &= Fs_t \\ \begin{bmatrix} w_t \\ r_t \end{bmatrix} &= Gs_t \end{aligned}$$

with F having all its eigenvalues on the unit circle.

Therefore, we begin by developing our optimization procedure in this well-known setting of asymptotic tracking, thereby obtaining alternative formulations in the discrete-time case. Using a very short and simple proof, we are able to give a complete solution to the problem of finding all *universal tracking regulators*, i.e., all regulators which achieve asymptotic tracking (1.13) for all values of the complex vector amplitudes $\{w^{(j)}\}_{j\in\mathcal{J}_w}$ and $\{r^{(j)}\}_{j\in\mathcal{J}_r}$, and which do not not depend on these amplitudes. This will be done in Section 4. As a preliminary for this, and to set up notations, in Section 3 we first consider an undisturbed system ($w_t \equiv 0$), and we characterize all regulators (1.12) achieving the design objective (1.13) for *all* reference signals r_t, not only harmonic ones, and *all* initial conditions; we shall refer to this property as *T-universal*. The solution of this problem is certainly known, but we include it for conceptual reasons.

However, if $\mu > k$, i.e., the dimension of r_t is larger than the number of outputs available for feedback, no universal tracking regulator exists, so a nonzero tracking error remains. To damp this error we turn to our main problem, namely to characterize all optimal universal regulators, as defined above. Also, we may want to use a criterion (1.8) even if asymptotic tracing is possible, if it is desirable to damp the control energy and/or some particular internal system variables. This is the topic of Section 5, where optimality in the linear class $\mathfrak{L}$ is studied. In Section 6 we show that these linear universal regulators are optimal also in the wider class of nonlinear regulators satisfying (1.11), provided slightly stronger technical conditions are satisfied. The complete solution is given. We note that a similar but different optimization problem, over a finite horizon, is considered in [26].

Obviously there is no a priori guarantee that a regulator which minimizes (1.8) will also satisfy other design specifications, and hence we look for complete solutions with many free parameters which then can be tuned by loop shaping. In fact, all our results are based on a parameterization derived in Section 2, which is akin to that of Youla and Kučera and which generalizes some parameterizations previously presented in [21, 22].

Finally, in Section 7, we give some simple numerical examples.

2. Linear stabilizing and realizable regulators

In order to design universal regulators we need a parameterization of all linear regulators

$$M(\sigma)u_t = N(\sigma)y_t + L(\sigma)r_t \tag{2.1}$$

which stabilize the control system (1.1) and which are realizable in a sense to be defined shortly. As before, σ is the backward shift $\sigma y_t = y_{t+1}$, and $M(\lambda)$, $N(\lambda)$ and $L(\lambda)$ are real matrix polynomials of dimensions $k \times k$, $k \times m$ and $k \times \mu$ respectively.

Let us consider a bit closer the meaning of (2.1) being stabilizing. To this end, note that the transfer functions Ψ_x, Ψ_u, Ψ_y from Ew_t to x_t, u_t and y_t respectively in the closed-loop system (1.1), (2.1) satisfy

$$(\lambda I_n - A)\Psi_x = B\Psi_u + I_n \tag{2.2a}$$
$$M(\lambda)\Psi_u = N(\lambda)\Psi_y \tag{2.2b}$$
$$\Psi_y = C\Psi_x \tag{2.2c}$$

so, in particular,

$$\Xi(\lambda)\begin{bmatrix}\Psi_x\\ \Psi_u\end{bmatrix} = \begin{bmatrix}I_n\\ 0\end{bmatrix}, \tag{2.3}$$

where $\Xi(\lambda)$ is the $(n+k)\times(n+k)$ matrix polynomial

$$\Xi(\lambda)=\begin{bmatrix}\lambda I_n - A & -B\\ -N(\lambda)C & M(\lambda)\end{bmatrix}. \tag{2.4}$$

Similarly, the transfer functions $\hat\Psi_x$, $\hat\Psi_u$ from r_t to x_t and u_t respectively are given by

$$\Xi(\lambda)\begin{bmatrix}\hat\Psi_x\\ \hat\Psi_u\end{bmatrix}=\begin{bmatrix}0\\ L(\lambda)\end{bmatrix}, \tag{2.5}$$

which together with (2.3) yields

$$\Psi:=\begin{bmatrix}\Psi_x & \hat\Psi_x\\ \Psi_u & \hat\Psi_u\end{bmatrix}=\Xi(\lambda)^{-1}\begin{bmatrix}I_n & 0\\ 0 & L(\lambda)\end{bmatrix}. \tag{2.6}$$

We shall say that the regulator (2.1) is *stabilizing* if the matrix polynomial $\Xi(\lambda)$ is *stable* i.e., $\det\Xi(\lambda)\neq 0$ for $|\lambda|\geq 1$.

Next we consider the condition that the regulator be realizable. Clearly (2.1) must be nonanticipatory in the sense that u_t does not depend on future values of y_t and r_t. To insure this, we must assume that

$$M(\lambda)^{-1}N(\lambda) \text{ and } M(\lambda)^{-1}L(\lambda) \text{ are proper,} \tag{2.7}$$

requiring in particular that $\det M(\lambda)\not\equiv 0$.

Let us investigate what properties Ψ must have for (2.7) to be satisfied. To this end, let us introduce the rational transfer functions

$$W_y(\lambda)=C(\lambda I_n - A)^{-1}B, \quad W_z(\lambda)=H(\lambda I_n - A)^{-1}B+J \tag{2.8}$$

from the control signal u_t to the outputs y_t and z_t respectively. Then it is easy to see that

$$\Psi_u=(M-NW_y)^{-1}NC(\lambda I_n - A)^{-1}, \quad \hat\Psi_u=(M-NW_y)^{-1}L \tag{2.9}$$

and that

$$\Psi_x=(\lambda I_n - A)^{-1}(B\Psi_u+I_n), \quad \hat\Psi_x=(\lambda I_n - A)^{-1}B\hat\Psi_u. \tag{2.10}$$

Writing (2.9) in the alternative form

$$\Psi_u=(I_n-M^{-1}NW_y)^{-1}M^{-1}NC(\lambda I_n - A)^{-1}, \quad \hat\Psi_u=(I_n-M^{-1}NW_y)^{-1}M^{-1}L,$$

we see that (2.7) implies that Ψ_u is strictly proper and $\hat\Psi_u$ is proper. In fact, $M^{-1}NW_y$ is strictly proper, making $I_n-M^{-1}NW_y$ as well as its inverse proper. Then, it follows from (2.10) that Ψ_x and $\hat\Psi_x$ are strictly proper also. Consequently,

$$\Psi(\infty)=\begin{bmatrix}0 & 0\\ 0 & \hat\Psi_u(\infty)\end{bmatrix} \quad \text{where } \hat\Psi_u(\infty) \text{ is finite} \tag{2.11}$$

so that x_t and u_t depend on w_s for $s<t$ only and on r_s for $s\leq t$ only. We shall say that the regulator (2.1) is *realizable* if condition (2.11) is satisfied. In the end of this section we shall demonstrate that any stabilizing and realizable regulator satisfies (2.7) so that the nonanticipatory property is implied (Corollary 2.3).

We say that two regulators

$$M_1(\sigma)u_t=N_1(\sigma)y_t+L_1(\sigma)r_t \quad \text{and} \qquad M_2(\sigma)u_t=N_2(\sigma)y_t+L_2(\sigma)r_t$$

are *equivalent* if there are stable $k \times k$ matrix polynomials Θ_1 and Θ_2 such that

$$[M_2, N_2, L_2] = \Theta_2\Theta_1^{-1}[M_1, N_1, L_1]. \tag{2.12}$$

Hence we allow the systems matrices M, N, L to have stable common factors, as coprimeness is not required. Clearly, as can be seen from (2.9) and (2.10), Ψ_x, Ψ_u, $\hat{\Psi}_x$ and $\hat{\Psi}_u$ are invariant under this equivalence, and so are the regulator transfer functions (2.7).

From now on, we assume that A is a stable matrix, i.e., $\det(\lambda I_n - A) \neq 0$ for all $|\lambda| \geq 1$. Since (A, B) is stabilizable and (C, A) is detectable, this is no restriction. In fact, it is well-known that the system (1.1) can be replaced by a similar system having a stable A-matrix but, in general, a larger dimension. (See any standard text, such as [1, 18].) Only under special conditions [15], including the case of complete state observation, is it possible to do this by constant feedback, but the system can always be stabilized by a dynamic observer. Then, extending the state space by including this observer, a system with stable A-matrix is obtained. For these reasons we shall from now on, without loss of generality, assume that A in (1.1) is a stable matrix.

The following theorem, generalizing a similar result in [22], provides a parameterization akin to the well-known Youla-Kučera parameterization. (We note that, if A is not stable, also the latter parameterization requires an observer-based prestabilization, increasing the dimension of the regulator; see, e.g., [32, p. 226].)

Theorem 2.1. *Let A be a stable matrix with $\chi(\lambda) := \det(\lambda I_n - A)$ being its characteristic polynomial, and let $G(\lambda)$ and $V_y(\lambda)$ be the matrix polynomials*

$$G(\lambda) = \chi(\lambda)(\lambda I_n - A)^{-1}, \quad V_y(\lambda) = CG(\lambda)B. \tag{2.13}$$

Moreover, let $\rho(\lambda)$ be an arbitrary stable scalar polynomial and let $R(\lambda)$ and $L(\lambda)$ be arbitrary matrix polynomials, of dimensions $k \times m$ and $k \times \mu$ respectively, such that

$$\deg(RCG) < \deg \rho, \quad \deg L \leq \deg \rho. \tag{2.14}$$

Then the regulator

$$M(\sigma)u_t = N(\sigma)y_t + L(\sigma)r_t \tag{2.15}$$

with

$$M(\lambda) = \rho(\lambda)I_k + R(\lambda)V_y(\lambda), \quad N(\lambda) = \chi(\lambda)R(\lambda), \tag{2.16}$$

is stabilizing and realizable, and for this regulator

$$\Psi_u(\lambda) = \frac{R(\lambda)}{\rho(\lambda)}CG(\lambda), \quad \hat{\Psi}_u(\lambda) = \frac{L(\lambda)}{\rho(\lambda)} \tag{2.17}$$

and

$$\det \Xi(\lambda) = \chi(\lambda)\,[\rho(\lambda)]^k \tag{2.18}$$

where Ξ is given by (2.4)*. Conversely, any stabilizing and realizable regulator* (2.15) *is equivalent to one constructed in this way.*

Before turning to the proof of this parameterization, let us briefly explain the nature of relation (2.18). Although $\chi(\lambda)$ is a factor in $\det \Xi(\lambda)$ for the regulator defined via (2.16), this is in general not the case for an arbitrary regulator belonging the same equivalence class. In fact, while the closed-loop transfer function Ψ and the regulator transfer functions $M^{-1}N$ and $M^{-1}L$ are invariant under the equivalence (2.12), Ξ is not. Taking the Schur complement, it immediately follows from (2.4) that

$$\det \Xi(\lambda) = \det(\lambda I_n - A) \det[M(\lambda) - N(\lambda)W_y(\lambda)], \tag{2.19}$$

where W_y is given by (2.8). Since, in general, the second factor is not a polynomial, χ is of course not a factor in $\det \Xi$ in general. Nevertheless, it will turn out to be useful to represent each equivalence class by a regulator that has this property.

Proof of Theorem 2.1. In view of (2.16), we have

$$M(\lambda) - N(\lambda)W_y(\lambda) = \rho(\lambda)I_k, \tag{2.20}$$

and consequently (2.17) follows from (2.9) and (2.18) follows from (2.19). By construction, therefore, $\Xi(\lambda)$ is a stable matrix polynomial, establishing that the regulator is stabilizing. Moreover, in view of (2.14), Ψ_u is strictly proper and $\hat{\Psi}_u$ is proper, i.e., $\Psi_u(\infty) = 0$ and $\hat{\Psi}_u(\infty)$ is finite. It then follows from (2.10) that Ψ_x and $\hat{\Psi}_x$ are strictly proper, and hence the regulator is realizable.

To prove the converse statement, suppose that $[M_0, N_0, L_0]$ is an arbitrary stabilizing and realizable regulator. Then (2.19) may be written

$$\det \Xi(\lambda) = \chi(\lambda)^{1-k} \det P(\lambda),$$

where $P(\lambda)$ is the $k \times k$ matrix polynomial

$$P := \chi M_0 - N_0 V_y = \chi(M_0 - N_0 W_y), \tag{2.21}$$

which is stable and full rank, since $\det P = \chi^{k-1} \det \Xi$ is stable and nontrivial. It follows from (2.9) that

$$P\Psi_u = N_0 C G, \quad P\hat{\Psi}_u = \chi L_0, \tag{2.22}$$

where Ψ_u and $\hat{\Psi}_u$ are the closed-loop transfer functions corresponding to the regulator $[M_0, N_0, L_0]$. Therefore, setting

$$\rho := \det P, \quad R := P_a N_0 = \rho P^{-1} N_0 \quad \text{and} \quad L := \chi P_a L_0 = \chi \rho P^{-1} L_0,$$

where $P_a := P^{-1} \det P$ is the adjoint matrix polynomial of P, (2.22) shows that Ψ_u and $\hat{\Psi}_u$ are given by (2.17). Since $[M_0, N_0, L_0]$ is a realizable regulator, it follows from (2.11) that the the degree conditions (2.14) hold. Consequently, defining M and N via (2.16), it follows from the first part of the theorem that $[M, N, L]$ is a stabilizing and and realizable regulator with the same closed-loop transfer functions Ψ_u and $\hat{\Psi}_u$ as $[M_0, N_0, L_0]$. It remains to show that $[M, N, L]$ and $[M_0, N_0, L_0]$ are equivalent. To this end, note that

$$N = \chi R = \chi \rho P^{-1} N_0.$$

Also it follows from (2.21) that

$$\chi \rho P^{-1} M_0 = \rho I_n + \rho P^{-1} N_0 V_y = \rho I_n + R V_y = M.$$

Consequently,

$$[M, N, L] = \chi\rho P^{-1}[M_0, N_0, L_0],$$

i.e., $[M, N, L]$ and $[M_0, N_0, L_0]$ are equivalent as required. $\square$

If $C = I_n$ so that $y_t = x_t$, the representation of stabilizing regulators can be simplified considerably, since ρ and R can be chosen so that cancellations occur. Since this formulation has a different form and, moreover, will be used later, we state it as a corollary. Note that, in view of the converse statement, this corollary is strictly speaking *not* a special case of Theorem 2.1. It is in fact a generalization of Lemma 4.3 in [21], but the proof here is new.

Corollary 2.2. *Let A be a stable matrix, and suppose that $C = I_n$. Let $\rho(\lambda)$ be an arbitrary real scalar stable polynomial, and let $R(\lambda)$ and $L(\lambda)$ be arbitrary real matrix polynomials, of dimensions $k \times n$ and $k \times \mu$ respectively, such that*

$$\deg R < \deg \rho \quad \deg L \leq \deg \rho. \tag{2.23}$$

Then the regulator

$$M(\sigma)u_t = N(\sigma)x_t + L(\sigma)r_t \tag{2.24}$$

with

$$M(\lambda) = \rho(\lambda)I_k + R(\lambda)B, \quad N(\lambda) = R(\lambda)(\lambda I_n - A) \tag{2.25}$$

is stabilizing and realizable, and, for this regulator,

$$\Psi_u(\lambda) = \frac{R(\lambda)}{\rho(\lambda)}, \quad \hat{\Psi}_u(\lambda) = \frac{L(\lambda)}{\rho(\lambda)}, \tag{2.26}$$

and $\det \Xi$ *satisfies* (2.18). *Conversely, any stabilizing and realizable regulator* (2.24) *is equivalent to one constructed in this way.*

Proof. Let the polynomials ρ and R be chosen as in the statement of the corollary, and take $\rho_0(\lambda) := \rho(\lambda)\chi(\lambda)$ and $R_0(\lambda) := R(\lambda)(\lambda I_n - A)$ to be the corresponding polynomials in Theorem 2.1. Then, since $C = I_n$ and $(\lambda I_n - A)G(\lambda) = \chi(\lambda)I_n$, the degree conditions (2.14) are satisfied for ρ_0 and R_0. Moreover, the corresponding regulator polynomials matrices (2.16), which we denote M_0 and N_0, become $M_0 = \chi M$ and $N_0 = \chi N$, where M and N are given by (2.25). Then, setting $L_0 = \chi L$, the regulator $[M_0, N_0, L_0]$ is stabilizing and and realizable by Theorem 2.1. Thanks to cancellation, therefore, $[M, N, L]$ is a stabilizing and realizable regulator for the problem of Corollary 2.2, as claimed.

Conversely, by Theorem 2.1, any stabilizing and realizable regulator (2.24) is equivalent to some regulator $[M, N, L]$ of the type described in Theorem 2.1, where we set $C = I_n$ everywhere. It remains to show that $[M, N, L]$ is also a regulator of the type described in the corollary. To this end, define $\hat{R} := RG$. This implies that $\chi R = \hat{R}(\lambda I_n - A)$, and hence the equations of Theorem 2.1 become those of the corollary with R replaced by $\hat{R}$. Hence $[M, N, L]$ is also a regulator in the sense of the corollary. $\square$

In the beginning of this section we demonstrated that the realizability condition (2.11) is a consequence of nonanticipatory condition (2.7). Next we show that the converse is also true, provided C has full rank as assumed in (1.6).

Corollary 2.3. *Suppose that* rank $C = m$. *Then, for any stabilizing regulator* (2.15), *the realizability condition* (2.11) *and the nonanticipatory condition* (2.7) *are equivalent.*

Proof. The proof is immediate in the special case $C = I_n$. In fact, for a regulator (2.24) with M and N given by (2.25), condition (2.7) is a direct consequence of the degree condition (2.23). For any other stabilizing regulator (2.24), it follows from the definition of equivalence.

The general case follows from the fact that (2.15) is a subclass of (2.24). In fact, writing (2.15) as

$$M(\sigma)u_t = N(\sigma)Cx_t + L(\sigma)r_t,$$

it follows from what has already been proved that $M^{-1}NC$ is proper. Since C has full rank, this implies that $M^{-1}N$ is proper. That $M^{-1}L$ is proper follows directly. □

3. T-universal regulators

As a preliminary for the analysis in Sections 4 and 5, in this section we consider the problem of controlling the undisturbed system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t && (3.1a)\\ y_t &= Cx_t && (3.1b)\\ z_t &= Hx_t + Ju_t && (3.1c) \end{aligned}$$

by feedback from the output y_t so that it tracks a given reference signal r_t in the sense that

$$|z_t - r_t| \to 0 \quad \text{as } t \to \infty. \tag{3.2}$$

As explained in Section 2 it is no restriction to assume that A is stable if it is assumed that (A, B) is stabilizable and (C, A) is detectable. The solution of this problem is simple and certainly known, but we include it for completeness and for conceptual reasons.

More precisely, we want to find a stabilizing and realizable regulator of the form

$$M(\sigma)u_t = N(\sigma)y_t + L(\sigma)r_t \tag{3.3}$$

which is *universal* for the asymptotic tracking problem in the sense that (3.2) holds for *all* solutions of (3.1), (3.3) and *all* reference signals r_t. More specifically we shall refer to this property as *T-universal.*

Clearly, for (3.3) to be stabilizing and realizable, the matrix polynomials $M(\lambda)$, $N(\lambda)$ and $L(\lambda)$ must satisfy the specifications of Theorem 2.1. It remains to investigate under what conditions the tracking criterion (3.2) is satisfied and under what conditions this regulator is T-universal.

We begin by deriving a necessary condition for T-universality. Consider a reference signal of the type

$$r_t = \mathrm{Re}\{\tilde{r}e^{i\theta t}\}, \tag{3.4}$$

where $\tilde{r} \in \mathbb{C}^\mu$ and $\theta \in \mathbb{R}$ are fixed but arbitrary. Then the closed-loop system (3.1), (3.3) has solutions

$$x_t = \mathrm{Re}\{\tilde{x}e^{i\theta t}\}, \quad y_t = \mathrm{Re}\{\tilde{y}e^{i\theta t}\}, \quad z_t = \mathrm{Re}\{\tilde{z}e^{i\theta t}\}, \quad u_t = \mathrm{Re}\{\tilde{u}e^{i\theta t}\} \tag{3.5}$$

with

$$\tilde{x} = W_x(\lambda)\tilde{u}, \quad \tilde{y} = W_y(\lambda)\tilde{u}, \quad \tilde{z} = W_z(\lambda)\tilde{u}, \tag{3.6}$$

where $\lambda = e^{i\theta}$, $W_x(\lambda) = (\lambda I_n - A)^{-1}B$, and W_y and W_z are defined by (2.8). Moreover,

$$\tilde{u} = \hat{\Psi}_u(\lambda)\tilde{r}. \tag{3.7}$$

But the tracking condition (3.2) requires that

$$|z_t - r_t| = |\mathrm{Re}\{(\tilde{z} - \tilde{r})e^{i\theta t}\}| \to 0 \quad \text{as } t \to \infty,$$

and, since θ is arbitrary, this implies that $\tilde{z} = \tilde{r}$. Therefore, it follows from (3.5) and (3.7) that

$$W_z(e^{i\theta})\hat{\Psi}_u(e^{i\theta})\tilde{r} = \tilde{r}. \tag{3.8}$$

Now, in order that the regulator (3.3) be T-universal, (3.8) must hold for all r_t, that is, for all $\tilde{r}$ and θ. Consequently, we must have

$$W_z(\lambda)\hat{\Psi}_u(\lambda) = I_\mu \tag{3.9}$$

on the unit circle and, by analytic continuation, in the rest of the complex plane.

Lemma 3.1. *A stabilizing and realizable regulator* (3.3) *is T-universal if and only if the identity* (3.9) *holds.*

Proof. We have already proved that (3.9) is a necessary condition for (3.3) to be T-universal, so it remains to prove that it is also sufficient. To this end, first assume that there are positive numbers M, ρ_0 such that $|r_t| \leq M\rho_0^t$ for all t. Then r_t has a $\mathcal{Z}$-transform

$$\tilde{r}(\lambda) = \sum_{t=0}^{\infty} r_t \lambda^{-t}$$

which converges for $|\lambda| > \rho_0$. It follows from (3.6) and (3.7) that $W_z(\lambda)\hat{\Psi}_u(\lambda)$ is the transfer function from r_t to z_t, and hence (3.1),(3.3) has a solution z_t with a $\mathcal{Z}$-transform $W_z(\lambda)\hat{\Psi}_u(\lambda)\tilde{r}(\lambda)$. But, if (3.9) holds, then $\tilde{z} = \tilde{r}$ and hence $|z_t - r_t| = 0$ for all t. Because of stability any other solution z_t tends asymptotically to this solution, and therefore (3.2) holds. If r_t increases so fast that it does not have a $\mathcal{Z}$-transform, set $r_t^N := r_t$ for $t = 0, 1, \dots, N$ and $r_t^N := 0$ for $t > N$, and let z_t^N be the corresponding z-solution. Then it is easy to see that $z_t^N = z_t$ for $t = 0, 1, \dots, N$. Since N is arbitrary, the conclusion follows. $\square$

As a corollary we see that $\hat{\Psi}_u(\infty)$ must be full rank, or else (3.9) will be violated. This implies that there are no delays between r_t and u_t. Indeed, the condition (3.9) for T-universality imposes some rather stringent conditions on the system (3.1). In particular, since W_z is $\mu \times k$ and $\hat{\Psi}_u$ is $k \times \mu$, (3.9) implies that $k \geq \mu$, and $J = W_z(\infty)$ must have full rank.

Theorem 3.2. *Suppose that A is stable. Then there exists a T-universal regulator for the tracking problem if and only if there is a proper rational $k \times \mu$ matrix function $X(\lambda)$ with no poles in the region $|\lambda| \geq 1$ which satisfies the equation*

$$W_z(\lambda)X(\lambda) = I_\mu, \tag{3.10}$$

which, in particular, implies that $k := \dim u_t \geq \mu := \dim r_t$*. In this case, let* ρ *be a stable scalar polynomial such that*

$$L(\lambda) = \rho(\lambda)X(\lambda) \tag{3.11}$$

is a matrix polynomial, and let $R(\lambda)$ *be a* $k \times m$ *matrix polynomial satisfying the first degree constraint* (2.14)*. Then, the regulator* (2.15)*, with* M *and* N *given by* (2.16)*, is a T-universal regulator for the tracking problem, and any other T-universal regulator is equivalent to one obtained in this way.*

Proof. First, suppose that there exists a T-universal regulator of the form (3.3). Then, according to Lemma 3.1, there exists a solution $X(\lambda)$ to (3.10) with the prescribed properties, namely $\hat{\Psi}_u(\lambda)$. In fact, in view of (2.9), (2.19) and the fact that $\Xi(\lambda)$ is stable, it follows that $\hat{\Psi}_u(\lambda)$ has no poles in the region $|\lambda| \geq 1$. Moreover, since the regulator is realizable, $\hat{\Psi}_u(\lambda)$ is proper.

Next, suppose that (3.10) has a solution $X(\lambda)$ which is proper with no poles in the region $|\lambda| \geq 1$, and let ρ, R and L be defined as in the theorem. (Note that in order to satisfy the first of degree conditions (2.14) we may need to choose ρ and L which are not coprime.) Then, by Theorem 2.1, the regulator (2.15) with M, N given by (2.16) is stabilizing and realizable and

$$\hat{\Psi}_u(\lambda) = \frac{L(\lambda)}{\rho(\lambda)}, \tag{3.12}$$

i.e., in view of (3.11), $\hat{\Psi}_u = X$. Consequently, it follows from (3.10) and Lemma 3.1 that the regulator is T-universal.

It remains to prove the last statement of the theorem. To this end, suppose that the regulator

$$M_0(\sigma)u_t = N_0(\sigma)y_t + L_0(\sigma)r_t \tag{3.13}$$

is T-universal. Then, in particular, it is stabilizing and realizable, and thus, by Theorem 2.1, there are some ρ, R and L with the properties specified in Theorem 2.1 such that the regulator (2.15) with M, N given by (2.16) is equivalent to (3.13). Now, $\hat{\Psi}_u$ is invariant under this equivalence. Therefore, since (3.9) holds for the regulator (3.13) by Lemma 3.1, (3.9) also holds for (2.15). However, by Theorem 2.1, (3.12) holds, and hence there is an X, namely $\hat{\Psi}_u$, satisfying (3.10) and (3.11). □

In general, a solution to (3.10) cannot be expected to be unique, but if $k = \mu$, only one solution is possible, namely

$$X(\lambda) = W_z^{-1}(\lambda),$$

and this would require that $W_z^{-1}(\lambda)$ is a stable, proper rational function, implying that W_z must be minimum phase with no zeros at infinity. In particular, $J := W_z(\infty)$ must be nonsingular.

Corollary 3.3. *Suppose that* A *is stable and the transfer function* W_z *is square, i.e.,* $k := \dim u_t = \mu := \dim r_t$*. Then there is a T-universal regulator for the tracking problem if and only if* W_z^{-1} *is proper with no poles in the region* $|\lambda| \geq 1$*. In this case, let* $\rho(\lambda)$ *be a stable scalar polynomial such that* $\rho(\lambda)W_z^{-1}(\lambda)$ *is a matrix polynomial* *and $R(\lambda)$ is a $k\times m$ matrix polynomial satisfying the degree requirement* (2.14)*. Then, if M and N are defined by* (2.16) *and L by*

$$L(\lambda) = \rho(\lambda)W_z^{-1}(\lambda) \tag{3.14}$$

the regulator (2.15) *is a T-universal regulator, and any other T-universal regulator is equivalent to one obtained in this way.*

A T-universal regulator exists only under rather special conditions. However, if we restrict our attention to harmonic reference signals (1.4), these conditions can be considerably relaxed, and we may also allow for external harmonic disturbances. This is the topic of the next section.

4. Universal tracking regulators in harmonically disturbed systems

We now return to the situation described in Section 1, where the control system takes the form (1.1) with a harmonic disturbance (1.2), and where there is a harmonic reference signal (1.4). Although we may allow the index set $\mathcal{I}_w$ to be empty, for tracking we must take $\mathcal{I}_r \neq \emptyset$.

The first question to be answered is when it is possible to find a regulator (1.12) in $\mathfrak{L}$ such that

$$|z_t - r_t| \to 0 \quad \text{as } t \to \infty, \tag{4.1}$$

which is *universal* in the sense that (4.1) holds for all values of $\{w^{(j)}\}_{j\in\mathcal{I}_w}$ and $\{r^{(j)}\}_{j\in\mathcal{I}_r}$ and does not depend on these vector amplitudes. We shall refer to such a regulator as a *universal tracking regulator*. For convenience, in the sequel we use the notation

$$\lambda_j := e^{i\theta_j} \quad j = 1, 2, \dots, N. \tag{4.2}$$

Theorem 4.1. *Suppose that the matrix A is stable, and let $G(\lambda)$ and $V_y(\lambda)$ be the matrix polynomials defined by* (2.13)*. Moreover, let $W_z(\lambda)$ be the $\mu\times k$ matrix function defined by* (2.8) *and $F(\lambda)$ the $m\times\ell$ matrix polynomial*

$$F(\lambda) := CG(\lambda)E = \chi(\lambda)C(\lambda I_n - A)^{-1}E. \tag{4.3}$$

Then, for a universal tracking regulator to exist in $\mathfrak{L}$, it is necessary that the rank condition

$$\operatorname{rank} W_z(\lambda_j) = \mu := \dim r_t \quad \textit{for all } j \in \mathcal{I}_r \tag{4.4}$$

holds, and it is sufficient that both rank conditions (4.4) *and*

$$\operatorname{rank} F(\lambda_j) = \ell := \dim w_t \quad \textit{for all } j \in \mathcal{I}_w \tag{4.5}$$

hold. In particular, (4.4) *requires that $\mu \leq k := \dim u_t$, and* (4.5) *that $\ell \leq m := \dim y_t$. More precisely, let $\rho(\lambda)$ be an arbitrary stable scalar real polynomial, and let $R(\lambda)$ and $L(\lambda)$ be matrix polynomials, of dimensions $k\times m$ and $k\times\mu$ respectively, satisfying the degree requirements* (2.14) *and the interpolation conditions*

$$\begin{aligned} W_z(\lambda_j)R(\lambda_j)F(\lambda_j) &= -\rho(\lambda_j)H(\lambda_j I_n - A)^{-1}E \quad \textit{for } j \in \mathcal{I}_w && \text{(4.6a)}\\ W_z(\lambda_j)L(\lambda_j) &= \rho(\lambda_j)I_\mu \quad \textit{for } j \in \mathcal{I}_r. && \text{(4.6b)} \end{aligned}$$

Then, if M and N are given by (2.16)*, the regulator* (2.15) *is a universal tracking regulator, and any other universal tracking regulator* (2.15) *is equivalent to one obtained in this way.*

Proof. Whenever a linear stabilizing regulator is applied to the system (1.1), the process (x_t, u_t) tends exponentially to the harmonic steady-state solution

$$x_t = \sum_{j=1}^{N} x^{(j)} e^{i\theta_j t}, \quad u_t = \sum_{j=1}^{N} u^{(j)} e^{i\theta_j t}, \tag{4.7}$$

where

$$x^{(j)} = \Psi_x(\lambda_j) E w^{(j)} + \hat{\Psi}_x(\lambda_j) r^{(j)} \tag{4.8a}$$
$$u^{(j)} = \Psi_u(\lambda_j) E w^{(j)} + \hat{\Psi}_u(\lambda_j) r^{(j)} \tag{4.8b}$$

Ψ_x, Ψ_u, $\hat{\Psi}_x$ and $\hat{\Psi}_u$ being the closed-loop transfer functions defined in Section 2. In fact, for any regulator in $\mathfrak{L}$, $\Xi(\lambda)$, defined by (2.4), is a stable matrix polynomial. In the same way, in view of (1.1c), z_t tends exponentially to

$$z_t = \sum_{j=1}^{N} z^{(j)} e^{i\theta_j t}, \quad z^{(j)} = W_z(\lambda_j) u^{(j)} + H(\lambda_j I_n - A)^{-1} E w^{(j)}. \tag{4.9}$$

Now, the basic idea is that the tracking condition (4.1) is achieved precisely when the cost function (1.7) is zero. It is easy to see that

$$\Phi_0 = \sum_{j=1}^{N} |z^{(j)} - r^{(j)}|^2. \tag{4.10}$$

To see this, observe that, if f_t and g_t are two harmonic vector sequences

$$f_t = \sum_{j=1}^{N} f^{(j)} e^{i\theta_j t} \quad \text{and} \quad g_t = \sum_{j=1}^{N} g^{(j)} e^{i\theta_j t},$$

with $\{\theta_j\}$ distinct as in (1.3), and Q is an arbitrary matrix of appropriate dimensions, then

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} f_t^* Q g_t = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} f_t^* Q g_t = \sum_{j=1}^{N} f^{(j)*} Q g^{(j)}. \tag{4.11}$$

Moreover, in view of (4.8b) and (4.9),

$$z^{(j)} - r^{(j)} = [W_z(\lambda_j)\Psi_u(\lambda_j)E + H(\lambda_j I_n - A)^{-1}E] w^{(j)} + [W_z(\lambda_j)\hat{\Psi}_u(\lambda_j) - I_\mu] r^{(j)},$$

and consequently (4.10) equals zero for all values of $\{w^{(j)}\}_{j\in\mathfrak{J}_w}$ and $\{r^{(j)}\}_{j\in\mathfrak{J}_r}$ if and only if

$$W_z(\lambda_j)\Psi_u(\lambda_j)E = -H(\lambda_j I_n - A)^{-1}E \quad \text{for } j \in \mathfrak{J}_w \tag{4.12a}$$
$$W_z(\lambda_j)\hat{\Psi}_u(\lambda_j) = I_\mu \quad \text{for } j \in \mathfrak{J}_r. \tag{4.12b}$$

Theorem 2.1 states that the regulator (2.15) is stabilizing if M and N are defined by (2.16) for some stable scalar real polynomial $\rho(\lambda)$ and some real matrix polynomials

$R(\lambda)$ and $L(\lambda)$ satisfying (2.14) and that any other stabilizing and realizable regulator (2.15) is equivalent to one obtained in this way. Moreover,

$$\Psi_u(\lambda) = \frac{R(\lambda)}{\rho(\lambda)} CG(\lambda), \quad \hat{\Psi}_u(\lambda) = \frac{L(\lambda)}{\rho(\lambda)}, \tag{4.13}$$

which inserted into (4.12), yields precisely (4.6).

If the rank conditions (4.5) and (4.4) hold, the interpolation conditions (4.6) have a solution, and the general solution is

$$\begin{aligned} R(\lambda_j) &= W_z(\lambda_j)^*[W_z(\lambda_j)W_z(\lambda_j)^*]^{-1}[-\rho(\lambda_j)H(\lambda_j I_n - A)^{-1}E] \\ &\qquad \times [F(\lambda_j)^*F(\lambda_j]^{-1}F(\lambda_j)^* + \tilde{R}_j \quad \text{for } j \in \mathfrak{I}_w \\ L(\lambda_j) &= \rho(\lambda_j)W_z(\lambda_j)^*[W_z(\lambda_j)W_z(\lambda_j)^*]^{-1} + \tilde{L}_j \quad \text{for } j \in \mathfrak{I}_r, \end{aligned}$$

where, for $j = 1, 2, \dots, N$, $\tilde{R}_j$ and $\tilde{L}_j$ are arbitrary matrices such that $W_z(\lambda_j)^*\tilde{R}_jF(\lambda_j) = 0$ and $W_z(\lambda_j)^*\tilde{L}_j = 0$. Here the degree of the stable polynomial ρ is chosen sufficiently high to satisfy the degree constraints (2.14). On the other hand, the rank condition (4.4) is also necessary for the existence of a universal tracking regulator. In fact, since $\rho(\lambda)$ is stable, (4.12b) cannot hold if rank $W_z(\lambda_j) < \mu$ for some $j = 1, 2, \dots, N$. □

Remark 4.2. The two rank conditions (4.5) and (4.4) in Theorem 4.1, which of course can be stated in terms of zeros of certain transfer functions, have different status. If (4.4) is violated, the interpolation condition (4.6b) cannot hold, so there could be no universal tracking regulator. On the other hand, if (4.4) holds but (4.5) does not, interpolation condition (4.6a) could still be valid, as the rank of the right member could be less than ℓ. However, this is a nongeneric situation, and hence cannot be expected to occur in practice. In fact, if $\ell > m$ and $F(\lambda_j)F(\lambda_j)^* > 0$, the following equation must hold:

$$H(\lambda_j I_n - A)^{-1}E\{I_\ell - F(\lambda_j)^*[F(\lambda_j)F(\lambda_j)^*]^{-1}F(\lambda_j)\} = 0,$$

which will occur only on a lower-dimensional algebraic set in the parameter space.

Theorem 4.1 provides a complete solution of a problem studied in various degrees of generality in [4, 5, 6, 7, 8, 13, 16] and of course consistent with the solutions given there, although given in a different form and in continuous time. If $w_t \equiv 0$, rank condition (4.5) becomes void and only (4.4), a considerably weaker version of condition (3.10) in Section 3, remains. Hence, for universal tracking regulators to exist the condition $\mu \le k$ is necessary, and if there are external disturbances w_t, in practice, we must also have $\ell \le m$. Consequently, as also noted in [4, 7, 8, 13, 16], asymptotic tracking is only possible under certain specific conditions.

Remark 4.3. (*Internal Model Principle.*) The situation most often studied in the literature is when $z_t \equiv y_t$, i.e., $H = C$, $J = 0$ and $\mu = m$, and when the regulator (2.15) takes the form

$$u_t = M(\sigma)^{-1}N(\sigma)(y_t - r_t),$$

obtained by setting $L(\lambda) = -N(\lambda)$. We assume that the rank conditions (4.4) and (4.5) are satisfied so that $\ell \le m \le k$. For robustness it is desirable to include a model of the disturbance dynamics in the regulator. This is the *internal model principle*. Following [3], we replace the matrix fraction representation $M^{-1}N$ by the (reachable) matrix fraction representation PD^{-1} so that $ND = MP$. The harmonic dynamics is then included in the regulator dynamics by setting $D(\lambda) = \phi(\lambda)D_0(\lambda)$, where $\phi(\lambda) := \Pi_{j=0}^{N}(\lambda - \lambda_j)$ and $D_0(\lambda)$ is a stable matrix polynomial. Then, by (2.16),

$$0 = N(\lambda_j)D(\lambda_j) = M(\lambda_j)P(\lambda_j) = [\rho(\lambda_j)I_k + R(\lambda_j)V_y(\lambda_j)]P(\lambda_j), \quad j = 1, 2, \dots, N,$$

which, in view of the fact that $V_y = \chi W_z$, yields

$$R(\lambda_j) = -\frac{\rho(\lambda_j)}{\chi(\lambda_j)}P(\lambda_j)[W_z(\lambda_j)P(\lambda_j)]^{-1}, \quad j = 1, 2, \dots, N,$$

where we have assumed that W_zP has no zeros in the points $\lambda_1, \dots, \lambda_N$. (Otherwise we include a simple feedback loop to move the zeros.) These $R(\lambda_j)$ clearly satisfy the interpolation conditions (4.6). In fact, since $H = C$, $J = 0$ and $L(\lambda) = -N(\lambda)$, by (2.16), these can be written

$$\begin{aligned} W_z(\lambda_j)R(\lambda_j)F(\lambda_j) &= -\frac{\rho(\lambda_j)}{\chi(\lambda_j)}F(\lambda_j) \quad \text{for } j \in \mathfrak{I}_w \\ W_z(\lambda_j)R(\lambda_j) &= -\frac{\rho(\lambda_j)}{\chi(\lambda_j)}I_\mu \qquad\quad \text{for } j \in \mathfrak{I}_r. \end{aligned}$$

Consequently we see that the internal-model-principle regulators form a subclass of the ones considered above.

The rank condition (4.5) becomes void if rank $C = n$, which is equivalent to the case with complete state information, i.e., the case when $y_t \equiv x_t$. Then the formulas for the regulator also simplify considerably.

Theorem 4.4. *Suppose that $C = I_n$ so that $y_t \equiv x_t$. Moreover, suppose that A is stable and that condition* (1.6) *holds. Then, there exists a universal tracking regulator* (2.24) *in $\mathfrak{L}$ if and only if the rank condition* (4.4) *holds. In fact, let $\rho(\lambda)$ be a stable scalar real polynomial, and let $R(\lambda)$ and $L(\lambda)$ be matrix polynomials satisfying the degree constraints* (2.23) *and the interpolation conditions*

$$\begin{aligned} W_z(\lambda_j)R(\lambda_j)E &= -\rho(\lambda_j)H(\lambda_jI_n - A)^{-1}E \quad \textit{for } j \in \mathfrak{I}_w && (4.14a) \\ W_z(\lambda_j)L(\lambda_j) &= \rho(\lambda_j)I_\mu \qquad\qquad\qquad\quad \textit{for } j \in \mathfrak{I}_r. && (4.14b) \end{aligned}$$

Then, if M and N are given by (2.25), *the regulator* (2.24) *is a universal tracking regulator, any other universal tracking regulator* (2.24) *is equivalent to one obtained in this way.*

Proof. The proof follows the same lines as that of Theorem 4.1, except that (2.26) from Corollary 2.2 is used in lieu of (4.13). Since rank $E = \ell \leq n$, $(E^*E)^{-1}$ exists and (4.14a) can be solved. □

When $\mu > k$, there are no universal tracking regulators, and in order to damp the steady state tracking error we shall therefore next turn to an optimization procedure. This is the topic of the next section.

5. Linear-quadratic optimization for tracking and damping

We now return to the optimization problem stated in the introduction. In this section we consider only linear regulators. Later, in Section 6, we demonstrate that under slightly stronger technical conditions the optimal universal regulators presented here are actually optimal in the much larger class $\mathcal{N}$, which includes nonlinear regulators.

Let us recall that the problem under consideration is to control the disturbed system (1.1) by feedback from the output y_t so as to minimize the cost function

$$\Phi = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \{\Lambda_0(x_t, u_t) + |z_t - r_t|^2\}, \tag{5.1}$$

where $\Lambda_0(x, u)$ is the quadratic form defined by (1.9). Hence we may not only want to damp the tracking error, but also some internal systems variables. As before, both the disturbance w_t and the reference signal r_t are harmonic and given by (1.5), where only the frequencies are known. The optimization is performed over the class $\mathcal{L}$ of stabilizing and realizable linear regulators (1.12). The problem under consideration is (i) to find the conditions under which there are optimal regulators which are *universal* in the sense that they are optimal for all choices of the amplitudes of (1.5) and independent of these and (ii) to characterize the class of all such universal optimal regulators.

To address this problem, let us first take a closer look at the cost function (5.1). A straight-forward reformulation taking (1.1c) into consideration yields

$$\Phi = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \{\Lambda(x_t, u_t) - r_t^* H x_t - x_t^* H^* r_t - r_t^* J u_t - u_t^* J^* r_t + {r_t}^* r_t\} \tag{5.2}$$

where $\Lambda(x, u)$ is the real quadratic form

$$\Lambda(x, u) = \begin{pmatrix} x \\ u \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{5.3}$$

with the real matrices Q, S and R given by

$$Q = Q_0 + H^* H, \quad S = S_0 + H^* J, \quad R = R_0 + J^* J. \tag{5.4}$$

The quadratic form (5.3) need not be nonnegative definite but must of course satisfy some condition insuring that $\inf \Phi \neq -\infty$. As we shall see, a sufficient condition for this is the *strong frequency domain condition*, i.e., that there is a $\delta > 0$ such that

$$\Lambda(\tilde{x}, \tilde{u}) \geq \delta(|\tilde{x}|^2 + |\tilde{u}|^2) \tag{5.5}$$

for all $\tilde{x} \in \mathbb{C}^n$, $\tilde{u} \in \mathbb{C}^k$ satisfying

$$\lambda\tilde{x} = A\tilde{x} + B\tilde{u} \tag{5.6}$$

for all $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. It can be shown [21] that if this condition fails in a strong way, i.e. there are $\tilde{x}$, $\tilde{u}$ and λ, $|\lambda| = 1$, such that $\Lambda(\tilde{x}, \tilde{u}) < 0$, then there is an external disturbance w_t such that $\inf \Phi = -\infty$. In this section, however, we shall only need the *weak frequency domain condition* that (5.5) and (5.6) hold for $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_N$, defined as in (4.2).

Both of these conditions are invariant under the action of the feedback group

$$(A, B) \to (TAT^{-1} + TBK, TB),$$

where T is a nonsingular matrix and K is an arbitrary matrix of appropriate dimensions. Moreover, since A has no eigenvalues on the unit circle, the inverse

$$A_\lambda := (\lambda I_n - A)^{-1} \tag{5.7}$$

exists for all λ on the unit circle, and hence $\tilde{x} = A_\lambda B\tilde{u}$ so that $\Lambda(\tilde{x}, \tilde{u}) = \tilde{u}^*\Pi(\lambda)\tilde{u}$, where $\Pi(\lambda)$ is the Hermitian $k \times k$ matrix function

$$\Pi(\lambda) = \begin{bmatrix} A_\lambda B \\ I_k \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} A_\lambda B \\ I_k \end{bmatrix} \tag{5.8}$$

In this notation the strong frequency domain condition may be written

$$\Pi(\lambda) > 0 \quad \text{for all } \lambda \text{ on the unit circle} \tag{5.9}$$

and the weak one as

$$\Pi(\lambda_j) > 0 \quad \text{for } j = 1, 2, \dots, N. \tag{5.10}$$

We now state the main result of this section. It will be strengthened in Section 6, where we show that, under mild technical conditions, the optimal universal regulator in $\mathfrak{L}$ is also optimal in the wider class $\mathfrak{N}$.

Theorem 5.1. *Let $G(\lambda)$, $V_y(\lambda)$ and $F(\lambda)$ be the matrix polynomials defined by* (2.13) *and* (4.3). *Suppose that the matrix A is stable and that the weak frequency domain condition* (5.10) *holds, and suppose that*

$$\operatorname{rank} F(\lambda_j) = \ell \quad \textit{for all } j \in \mathfrak{I}_w, \tag{5.11}$$

i.e., in particular that $m := \dim y_t \geq \ell := \dim w_t$. Then, there exists an optimal regulator in $\mathfrak{L}$ which is universal in the sense that it is optimal for all values of $\{w^{(j)}\}_{j\in\mathfrak{I}_w}$ and $\{r^{(j)}\}_{j\in\mathfrak{I}_r}$ and does not depend on these vector amplitudes.

More precisely, let $\rho(\lambda)$ be an arbitrary stable scalar real polynomial, and let $R(\lambda)$ and $L(\lambda)$ be matrix polynomials, of dimensions $k\times m$ and $k \times \mu$ respectively, satisfying the degree requirements (2.14) *and the interpolation conditions*

$$\begin{aligned} R(\lambda_j)F(\lambda_j) &= \rho(\lambda_j)U(\lambda_j) \quad \textit{for } j \in \mathfrak{I}_w \qquad &(5.12a)\\ L(\lambda_j) &= \rho(\lambda_j)\hat{U}(\lambda_j) \quad \textit{for } j \in \mathfrak{I}_r \qquad &(5.12b)\end{aligned}$$

with U and $\hat{U}$ given by

$$U(\lambda) = -\Pi(\lambda)^{-1}[QA_\lambda B + S]^*A_\lambda E, \qquad \hat{U}(\lambda) = \Pi(\lambda)^{-1}W_z(\lambda)^*, \tag{5.13}$$

where $W_z(\lambda) := HA_\lambda B + J$. Then the regulator (2.15) *is an universal regulator, which is optimal in $\mathfrak{L}$, provided M and N are given by* (2.16), *and any other universal regulator* (2.15), *which is optimal in $\mathfrak{L}$, is equivalent to one obtained in this way.*

Since, by assumption, $F(\lambda_j)^*F(\lambda_j)$ is nonsingular for $j \in \mathcal{J}_w$, (5.12a) has the solution

$$R(\lambda_j) = \rho(\lambda_j)U(\lambda_j)[F(\lambda_j)^*F(\lambda_j)]^{-1}F(\lambda_j)^* + \tilde{R}_j, \quad \tilde{R}_jF(\lambda_j) = 0 \tag{5.14}$$

for $j \in \mathcal{J}_w$, and these are precisely all solutions of (5.12a). Clearly, there are always matrix polynomials $R(\lambda)$ and $L(\lambda)$ satisfying (5.14), (5.12b) and the degree constraints (2.14), provided the degree of the stable scalar polynomial $\rho(\lambda)$ is chosen sufficiently large.

Remark 5.2. If $m < \ell$, there exist optimal regulators, but, as explained in Remark 4.2, universality is not a generic property, and therefore, for all practical purposes, there are no optimal universal regulators if $m < \ell$.

Remark 5.3. Before proceeding to the proof of Theorem 5.1, let us make certain that it is consistent with the results of Section 4. To this end, let us consider a cost function (1.7), i.e., suppose that $\Lambda_0 = 0$. Then

$$\Pi(\lambda) = W_z(\lambda)^*W_z(\lambda),$$

where the $\mu \times k$ matrix function W_z is given by (2.8). If $\mu < k$, the weak frequency domain condition cannot hold, so Theorem 5.1 does not apply. Instead, Theorem 4.1 should be used. If $\mu = k$, the weak frequency domain condition is a consequence of condition (4.4), and it is easy to check that the optimal cost will be zero, as required by Theorem 4.1. Moreover, interpolation conditions (5.12) and (4.6) are identical. Finally, if $\mu > k$, no universal tracking regulator is exists by Theorem 4.1, and the optimal cost will be nonzero in general.

Remark 5.4. (*Generalized Internal Model Principle.*) As in Remark 4.3, let us consider the case when $z_t \equiv y_t$, so that $H = C$, $J = 0$, $\mu = m$ and $V_y = \chi W_z$, and $L(\lambda) = -N(\lambda)$ in the regulator (2.15). For simplicity, also assume that $\mathcal{J}_r = \mathcal{J}_w$. If $\Lambda_0 = 0$ and $m = \mu \geq k$ and $m \geq \ell$, the interpolation conditions (5.12) can be written

$$\begin{aligned} R(\lambda_j)F(\lambda_j) &= -\frac{\rho(\lambda_j)}{\chi(\lambda_j)}[W_z(\lambda_j)^*W_z(\lambda_j)]^{-1}W_z(\lambda_j)^*F(\lambda_j) \\ R(\lambda_j) &= -\frac{\rho(\lambda_j)}{\chi(\lambda_j)}[W_z(\lambda_j)^*W_z(\lambda_j)]^{-1}W_z(\lambda_j)^* \end{aligned}$$

for $j = 1, 2, \ldots, N$, as can be seen from (2.16), (5.13) and the fact that $Q = C^*C$, $S = 0$ and $R = 0$. All of these interpolation conditions are satisfied if the second set is, and in this case (2.16) implies that

$$M(\lambda_j) = 0 \quad j = 1, 2, \ldots, N,$$

which could be interpreted as a *generalized internal model principle* for the the optimization problem.

The basic idea behind the proof of Theorem 5.1 is, as for Theorem 4.1, that, whenever a linear stabilizing regulator is applied to the system (1.1), the process (x_t, u_t) tends exponentially to the harmonic steady-state solution (4.7). Therefore, the cost function (5.1) depends only on the harmonic component (4.7) of (x_t, u_t). In fact, we have the following lemma. The proof follows from a simple completion-of-squares argument and is deferred to Appendix A.

Lemma 5.5. *Let (x_t, u_t) be any solution to the closed loop system* (1.1), (1.12), *where* (1.12) *is a stabilizing and realizable regulator, and suppose that the weak frequency domain condition* (5.10) *holds. Then the cost function* (5.1) *exists as a usual limit, and it takes the value*

$$\Phi = \sum_{j=0}^{N} \{(u^{(j)} - u^{(j)}_{opt})^* \Pi(\lambda_j)(u^{(j)} - u^{(j)}_{opt})\} + \Phi_{min}, \tag{5.15}$$

where, for $j = 1, 2, \ldots, N$,

$$u^{(j)}_{opt} = U(\lambda_j) w^{(j)} + \hat{U}(\lambda_j) r^{(j)} \tag{5.16}$$

with U and $\hat{U}$ given by (5.13) *and Φ_{min} by*

$$\Phi_{min} = \sum_{j=1}^{N} \Phi^{(j)}_{min}, \qquad \Phi^{(j)}_{min} = q_j - u^{(j)*}_{opt} \Pi(\lambda_j) u^{(j)}_{opt}, \tag{5.17}$$

where

$$\begin{aligned} q_j = {}& [A_{\lambda_j} E w^{(j)}]^* (Q - H^* H)[A_{\lambda_j} E w^{(j)}] \\ & + [H A_{\lambda_j} E w^{(j)} - r^{(j)}]^* [H A_{\lambda_j} E w^{(j)} - r^{(j)}]. \end{aligned} \tag{5.18}$$

In the expression (5.15) for the cost function Φ, only $u^{(1)}, u^{(2)}, \ldots, u^{(N)}$ depend on the regulator to be chosen. They are defined by (4.8b), i.e.,

$$u^{(j)} = \Psi_u(\lambda_j) E w^{(j)} + \hat{\Psi}_u(\lambda_j) r^{(j)}. \tag{5.19}$$

Recall that we consider the class $\mathcal{W}$ of external disturbances with arbitrary $w^{(j)}$ for $j \in \mathfrak{I}_w$ and $w^{(j)} = 0$ for $j \in \bar{\mathfrak{I}}_w = \{1, 2, \ldots, N\} \setminus \mathfrak{I}_w$ and the class $\mathcal{R}$ of reference signals with $r^{(j)}$ for $j \in \mathfrak{I}_r$ and $r^{(j)} = 0$ for $j \in \bar{\mathfrak{I}}_r = \{1, 2, \ldots, N\} \setminus \mathfrak{I}_r$.

Consequently, if we could find a stabilizing and realizable regulator (1.12) such that $u^{(1)}, u^{(2)}, \ldots, u^{(N)}$ satisfy the optimality conditions

$$u^{(j)} = u^{(j)}_{\text{opt}}, \quad j = 1, 2, \ldots, N, \tag{5.20}$$

which, in view of (5.19), is the same as

$$U(\lambda_j) w^{(j)} + \hat{U}(\lambda_j) \hat{w}^{(j)} = \Psi_u(\lambda_j) E w^{(j)} + \hat{\Psi}_u(\lambda_j) r^{(j)}, \tag{5.21}$$

then this regulator would be optimal. If, in addition, this regulator does not depend on the amplitudes $w^{(1)}, w^{(2)}, \ldots, w^{(N)}$ and $r^{(1)}, r^{(2)}, \ldots, r^{(N)}$ and the conditions (5.21) hold for all $\{w^{(j)}\}_{\mathfrak{I}_w}$ and $\{r^{(j)}\}_{\mathfrak{I}_r}$, i.e., all disturbances in $\mathcal{W}$ and all reference signals in $\mathcal{R}$, then this optimal regulator is also universal. This condition holds if and only if

$$\Psi_u(\lambda_j) E = U(\lambda_j) \quad \text{for } j \in \mathfrak{I}_w \tag{5.22a}$$

$$\hat{\Psi}_u(\lambda_j) = \hat{U}(\lambda_j) \quad \text{for } j \in \mathfrak{I}_r. \tag{5.22b}$$

Proof of Theorem 5.1. Theorem 2.1 states that the regulator (2.15) is stabilizing if M and N are defined by (2.16) for some stable scalar real polynomial $\rho(\lambda)$ and some real matrix polynomial $R(\lambda)$ satisfying (2.14) and that any other stabilizing and realizable regulator (2.15) is equivalent to one obtained in this way. Moreover,

$$\Psi_u(\lambda) = \frac{R(\lambda)}{\rho(\lambda)}CG(\lambda), \quad \hat{\Psi}_u(\lambda) = \frac{L(\lambda)}{\rho(\lambda)}. \tag{5.23}$$

We have demonstrated above that (5.22) is a necessary condition for the regulator (2.15) to be an optimal universal regulator, and inserting (5.23) into (5.22) yield precisely (5.12). Clearly, as we have already discussed, there are always matrix polynomials $R(\lambda)$ and $L(\lambda)$ satisfying these conditions and the degree constraints (2.14) provided the degree of the stable scalar polynomial $\rho(\lambda)$ is chosen sufficiently large, and provided condition (5.11) is satisfied.

It remains to prove the converse statement. For any optimal universal regulator $[\hat{M}, \hat{N}, \hat{L}]$, the value Φ of the cost function (5.1) equals $\Phi_{\min}$, defined by (5.17). It follows from (5.15) and the fact that $\Pi(\lambda_j) > 0$, for $j = 1, 2, \dots, N$, that (5.20) holds for all $\{w^{(j)}\}_{j\in\mathfrak{I}_w}$, and $\{r^{(j)}\}_{j\in\mathfrak{I}_r}$. Therefore (5.22) follows from (5.21). By Theorem 2.1, the regulator $[\hat{M}, \hat{N}, \hat{L}]$ is equivalent to (2.15) with M, N given by (2.16) for some ρ, R, L satisfying the requirements of Theorem 5.1. This regulator is also optimal since equivalent regulators have the same cost Φ. It is also universal because $[M, N, L]$ does not depend on $\{w^{(j)}\}_{j\in\mathfrak{I}_w}$ and $\{r^{(j)}\}_{j\in\mathfrak{I}_r}$. □

Corollary 5.6. *The optimal value of the cost function* (5.1) *in the class* $\mathfrak{L}$ *is* Φ_{min}*, defined by* (5.17) *and* (5.16)*.*

Note that, although an optimal universal regulator will not depend on $\{w^{(j)}\}_{j\in\mathfrak{I}_w}$ and $\{r^{(j)}\}_{j\in\mathfrak{I}_r}$, the cost function (5.17) will.

In the special case of complete state information, i.e., $y_t \equiv x_t$, condition (5.11) is always satisfied. In view of Corollary 2.2, Theorem 5.1 can be considerably simplified in this case, so we state it separately. The proof is the same as for Theorem 5.1, except that we now use the equations of Corollary 2.2.

Theorem 5.7. *Suppose that* $C = I_n$ *so that* $y_t \equiv x_t$*. Moreover, suppose that* A *is stable and that condition* (1.6) *holds. Then, if the weak frequency domain condition* (5.10) *holds, there exists a universal regulator* (2.24)*, which is optimal in* $\mathfrak{L}$*. In fact, let* $\rho(\lambda)$ *be a stable scalar real polynomial, and let* $R(\lambda)$ *and* $L(\lambda)$ *be matrix polynomials satisfying the degree constraints* (2.23) *and the interpolation conditions*

$$\begin{aligned} R(\lambda_j)E &= \rho(\lambda_j)U(\lambda_j) \quad \text{for } j \in \mathfrak{I}_w & (5.24a) \\ L(\lambda_j) &= \rho(\lambda_j)\hat{U}(\lambda_j) \quad \text{for } j \in \mathfrak{I}_r, & (5.24b) \end{aligned}$$

where U *and* $\hat{U}$ *are defined as in* (5.13)*. Then, if* M *and* N *are given by* (2.25)*, the regulator* (2.24) *is a universal regulator, which is optimal in* $\mathfrak{L}$*. Conversely, any other universal regulator* (2.24)*, which is optimal in* $\mathfrak{L}$*, is equivalent to one obtained in this way. Finally, the optimal value of the cost function* (5.1) *is given by* (5.17)*.*

Since, by assumption, E^*E is a nonsingular matrix of dimension $\ell \times \ell$, (5.24a) has the solution

$$R(\lambda_j) = \rho(\lambda_j)U(\lambda_j)(E^*E)^{-1}E^* + \tilde{R}_j, \quad \tilde{R}_jE = 0 \tag{5.25}$$

for $j \in \mathcal{J}_w$. There are always matrix polynomials $R(\lambda)$ and $L(\lambda)$ satisfying (5.25), (5.24b) and the degree constraints (2.23) provided the degree of the stable scalar polynomial $\rho(\lambda)$ is chosen sufficiently large.

6. Optimality in the class of nonlinear regulators

In this section we show that the universal optimal linear regulators described in Theorems 5.1 and 5.7 are actually optimal in a wide class of nonlinear regulators. We now define this class.

Given the control system (1.1), consider the class $\mathcal{N}$ of nonlinear regulators

$$u_t = \varphi_t(y_t, y_{t-1}, \dots, y_{t-q}, r_t, r_{t-1}, \dots, r_{t-p}), \tag{6.1}$$

which are stabilizing in the sense that any solution (x_t, u_t) of the closed loop system consisting of (1.1) and (6.1) satisfies the condition

$$\frac{1}{\sqrt{t}}|x_t| \to 0 \quad \text{as } t \to \infty. \tag{6.2}$$

This stability condition is quite weak but will suffice for our purposes. Of course, a weaker condition has the advantage of allowing for a larger class of controls.

We consider the same problem as in Section 5, except that we now optimize over all regulators in $\mathcal{N}$. Clearly, $\mathcal{N} \supset \mathcal{L}$. The only price we have to pay for this generalization is that the weak frequency domain condition needs to be replaced by the strong one.

Theorem 6.1. *Let A be stable, and suppose that the rank condition* (5.11) *holds. Then, if the strong frequency domain condition* (5.9) *holds, the linear optimal universal regulators of Theorem* 5.1 *are optimal in the class* $\mathcal{N}$.

It turns out that Theorem 6.1 is a simple consequence of the corresponding result for complete state information. In fact, the class of stabilizing and realizable regulators

$$M(\sigma)u_t = N(\sigma)y_t + L(\sigma)r_t \quad \text{with } y_t = Cx_t$$

is a subclass of the class of stabilizing and realizable regulators

$$M(\sigma)u_t = N(\sigma)x_t + L(\sigma)r_t$$

in that only a special structure of N is required. But, as seen in Section 5, an optimal universal regulator in the former class is optimal also in the latter, since the same optimal value $\Phi_{\min}$ is achieved (Corollary 5.6 and Theorem 5.7). (The only difference between the cases of complete and incomplete state information is that a higher degree regulator may be required in the latter case to achieve the optimum.) Consequently, if we can prove the following theorem, we have also proved Theorem 6.1.

Theorem 6.2. *Let A be stable, and suppose that $C = I_n$ and that* $\operatorname{rank} E = \ell$. *Then, if the strong frequency domain condition* (5.9) *holds, the linear optimal universal regulators of Theorem* 5.7 *are optimal in the class* $\mathcal{N}$.

In order to prove this theorem we consider an optimization problem which unlike that in Section 5 does not require that a linear regulator has been applied. More precisely, let us first consider the problem of finding a process $\{(x_t, u_t)\}_{t\in\mathbb{Z}_+}$ which minimizes the cost function (1.8), subject to the constraints (6.2) and

$$x_{t+1} = Ax_t + Bu_t + v_t, \quad x_0 = a, \tag{6.3}$$

where now $\{r_t\}_{t\in\mathbb{Z}_+}$ and $\{v_t\}_{t\in\mathbb{Z}_+}$ are arbitrary bounded and complex-valued vector sequences.

It is well-known (see, e.g., [29, 25, 24, 20, 21]) that, if the strong frequency domain condition (5.9) holds and (A, B) is stabilizable, then the algebraic Riccati equation

$$P = A^*PA - (A^*PB + S)(B^*PB + R)^{-1}(A^*PB + S)^* + Q, \tag{6.4}$$

has a unique symmetric solution P which renders the feedback matrix

$$\Gamma = A + BK \quad \text{where } K = -(B^*PB + R)^{-1}(A^*PB + S)^* \tag{6.5}$$

stable in the sense that all eigenvalues of Γ lie strictly inside the unit circle. We shall refer to this solution as the *stabilizing solution* of (6.4). For this solution we also have that

$$\hat{R} = B^*PB + R \tag{6.6}$$

is positive definite.[1]

Then we have the following result, which should be compared to Theorem 2.3 in [21], and the proof of which we defer to Appendix B.

Lemma 6.3. *Let* (A, B) *be stabilizable and suppose that the strong frequency domain condition* (5.9) *holds so that* (6.4) *has a stabilizing solution* P*. Moreover, let*

$$\pi_t = -\hat{R}^{-1}(B^*p_{t+1} + B^*Pv_t - J^*r_t), \tag{6.7}$$

where

$$p_t = \sum_{k=t}^{\infty}(\Gamma^*)^{k-t+1}Pv_k - \sum_{k=t}^{\infty}(\Gamma^*)^{k-t}(H + JK)^*r_k. \tag{6.8}$$

Then the problem to minimize the cost function (1.8) *subject to constraints* (6.2) *and* (6.3) *is solved by a process* (x_t, u_t) *such that*

$$u_t = Kx_t + \pi_t + \epsilon_t, \tag{6.9}$$

where K *is given by* (6.5) *and* $\{\epsilon_t\}_{t\in\mathbb{Z}_+}$ *is any vector sequence such that*

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T}|\epsilon_t|^2 = 0. \tag{6.10}$$

The optimal value of the cost function is

$$\Phi_{min} = \limsup_{T\to\infty} \frac{1}{T}\left(-q_{T+1}\right), \tag{6.11}$$

where

$$q_{t+1} = q_t - v_t^*Pv_t - p_{t+1}^*v_t - v_t^*p_{t+1} + \pi_t^*\hat{R}\pi_t + r_t^*r_t, \quad q_0 = 0. \tag{6.12}$$

If the limit $\lim_{T\to\infty}\frac{1}{T}q_{T+1}$ *exists, any optimal process* (x_t, u_t) *is produced in this way.*

[1] Note that there is a misprint on page 788 in [21]: In Theorem 2.1, replace 'statements hold' for 'statements are equivalent'.

Note that the control (6.9) cannot in general be used in practice, since it depends on future values of v_k and r_k. Even in the harmonic case when this dependence can be resolved, this control law has serious disadvantages [21, Section III]. It is developed here as an instrument of proof.

Next, let us return to our original problem and take $v_t := Ew_t$ and r_t to be harmonic, given by (1.5). Then a simple calculation, using (6.7) and (6.8), yields the representation

$$\pi_t = \sum_{j=1}^{N} \pi^{(j)} e^{i\theta_j t} \quad \text{with } \pi^{(j)} = \Pi_j w^{(j)} + \hat{\Pi}_j r^{(j)}, \tag{6.13}$$

where

$$\begin{aligned} \Pi_j &= -\hat{R}^{-1} B^* (I - \lambda_j \Gamma^*)^{-1} PE, \\ \hat{\Pi}_j &= -\hat{R}^{-1} [J^* + \lambda_j B^* (I - \lambda_j \Gamma^*)^{-1} (H + JK)^*]. \end{aligned}$$

We are now in a position to prove Theorem 6.1.

Proof of Theorem 6.1. Clearly, for any regulators in $\mathcal{N}$, (6.11) is a lower bound for the cost Φ. Therefore, if we can demonstrate that there is a regulator in $\mathcal{L}$ which achieves the same value (6.11) of the cost Φ, this regulator must be optimal also in $\mathcal{N}$, and so must all regulators which are optimal in $\mathcal{L}$.

To this end, let us introduce a new control $\hat{u}_t$ so that

$$u_t = Kx_t + \hat{u}_t, \tag{6.14}$$

transforming the system (1.1a) to

$$x_{t+1} = \Gamma x_t + B\hat{u}_t + Ew_t. \tag{6.15}$$

We want to find a stabilizing and realizable regulator

$$M(\sigma)\hat{u}_t = N(\sigma)x_t + L(\sigma)r_t \tag{6.16}$$

so that the closed loop system (6.14), (6.15), (6.16) has a solution (x_t, u_t) satisfying (6.9) for some ϵ_t with the property (6.10). Then, by Lemma 6.3, the regulator (6.14), (6.16), i.e.,

$$M(\sigma)u_t = [N(\sigma) + M(\sigma)K]x_t + L(\sigma)r_t, \tag{6.17}$$

is optimal in $\mathcal{N}$. Therefore, the optimal linear regulators of Theorem 2.1 must be optimal also in $\mathcal{N}$.

Since (6.16) is stabilizing, the solution $(x_t, \hat{u}_t)$ of the closed-loop system (6.15), (6.16) tends exponentially to a harmonic solution

$$x_t^0 = \sum_{j=1}^{N} x^{(j)} e^{i\theta_j t}, \quad \hat{u}_t^0 = \sum_{j=1}^{N} \hat{u}^{(j)} e^{i\theta_j t},$$

which of course yields the same value to Φ as (x_t, u_t). Now, if we can choose M, N, L so that

$$\hat{u}^{(j)} = \pi^{(j)} \quad \text{for } j = 1, 2, \ldots, N, \tag{6.18}$$

and hence $\hat{u}_t^0 \equiv \pi_t$, then $\epsilon_t := \hat{u}_t - \pi_t$ has the the property (6.10), and (6.14) becomes (6.9) as required.

To show that there are M, N, L such that (6.18) holds, we first apply Corollary 2.2 to the system (6.15), where Γ takes the place of A and $\hat{u}_t$ that of u_t. In fact, by Corollary 2.2, there is a stable scalar polynomial ρ and matrix polynomials R, L such that $\deg R < \deg \rho$ and $\deg L \leq \deg \rho$ so that M, N are given by

$$M(\lambda) = \rho(\lambda)I_k + R(\lambda)B \quad \text{and} \quad N(\lambda) = R(\lambda)(\lambda I_n - \Gamma)$$

and

$$\Psi_{\hat{u}}(\lambda) = \frac{R(\lambda)}{\rho(\lambda)}, \quad \hat{\Psi}_{\hat{u}}(\lambda) = \frac{L(\lambda)}{\rho(\lambda)}.$$

But $\hat{u}_t$ tends exponentially to the harmonic solution $\hat{u}_t^0$. Since therefore

$$\hat{u}_t^0 = \sum_{j=1}^{N} \left[\frac{R(\lambda_j)}{\rho(\lambda_j)} E w^{(j)} + \frac{L(\lambda_j)}{\rho(\lambda_j)} r^{(j)} \right] e^{i\theta_j t}$$

and π_t is given by (6.13), the optimality condition (6.18) will be satisfied for all $\{w^{(j)}\}_{j \in \mathfrak{I}_w}$ and $\{r^{(j)}\}_{j \in \mathfrak{I}_r}$ if

$$\begin{aligned} R(\lambda_j)E &= \rho(\lambda_j)\Pi_j, \quad j \in \mathfrak{I}_w \\ L(\lambda_j) &= \rho(\lambda_j)\hat{\Pi}_j, \quad j \in \mathfrak{I}_r \end{aligned}$$

Since E is full rank, in view of the discussion in Section 5 ρ, R, L can be chosen to satisfy these interpolation conditions. $\square$

7. Some simple numerical examples

To illustrate the results of this paper, let us consider the system

$$\begin{cases} y_{t+2} + a y_{t+1} + b y_t = u_t + w_t \\ z_t = y_{t+1} + c u_t \end{cases} \tag{7.1}$$

where u_t is the control, y_t and z_t are outputs, and the characteristic polynomial

$$\chi(\lambda) = \lambda^2 + a\lambda + b$$

is stable with $b \neq 0$. Defining the state

$$x_t = \begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix},$$

the plant equations (7.1) can be written in the state space form (1.1), where

$$A = \begin{bmatrix} -a & -b \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so that χ is the characteristic polynomial of A, and

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad J = c.$$

The matrix polynomials (2.13) are

$$G(\lambda) = \begin{bmatrix} \lambda & -b \\ 1 & \lambda + a \end{bmatrix} \quad \text{and} \quad V_y(\lambda) = 1.$$

Let us first take $w_t \equiv 0$ and consider the problem to find a T-universal regulator

$$M(\sigma)u_t = N(\sigma)y_t + L(\sigma)r_t \tag{7.2}$$

so that z_t tends asymptotically to r_t. By Corollary 3.3, a T-universal regulator exists if and only if

$$c \neq 0 \quad \text{and} \quad \psi \text{ stable,} \tag{7.3}$$

where

$$\psi(\lambda) := c\lambda^2 + (ac+1)\lambda + bc.$$

In fact, $W_z(\lambda) = \psi(\lambda)/\chi(\lambda)$. In this case, (7.2) is a T-universal regulator if and only if

$$M = \rho_0\psi + R, \quad N = \chi R, \quad L = \rho_0\chi \tag{7.4}$$

for some polynomials ρ_0 and R such that ρ_0 is stable and $\deg R < \deg \rho_0 + 1$ or is equivalent to one obtained in this way. This corresponds to the choice $\rho = \rho_0\psi$. Of course asymptotic tracking is achieved for *all* choices of reference signal r_t.

If, instead, we consider a reference signal

$$r_t = \alpha_1 \cos(\theta_1 t + \varphi_1) + \alpha_2 \cos(\theta_2 t + \varphi_2), \tag{7.5}$$

where the frequencies θ_1, θ_2 are given, but the amplitudes α_1, α_2 and the phases φ_1, φ_2 are unknown, the class of regulators (7.2) which achieve asymptotic tracking is much larger, and condition (7.3) need not be satisfied but can be exchanged for

$$\psi(e^{i\theta_j}) \neq 0 \quad \text{for } j = 1, 2. \tag{7.6}$$

In fact, by Theorem 4.1, in this case we may choose any stabilizing regulator

$$[\rho(\sigma) + R(\sigma)]u_t = \chi(\sigma)R(\sigma)y_t + L(\sigma)r_t, \tag{7.7}$$

provided ρ is stable and the degree constraint (2.14) and the interpolation conditions

$$L(e^{i\theta_j}) = \rho(e^{i\theta_j})\chi(e^{i\theta_j})/\psi(e^{i\theta_j}) \quad \text{for } j = 1, 2$$

are satisfied. The same regulator is obtained by applying Theorem 5.1, now observing that (7.6) is the weak frequency domain condition; see Remark 5.3. This allows for more tuning parameters to satisfy other design specifications. Of course, if condition (7.3) is fulfilled, the T-universal regulator can still be used.

As a numerical example, suppose that $a = 0.4$, $b = 0.7$ and $c = 1$, and let $\theta_1 = 1.0$ and $\theta_2 = 0.5$. Then condition (7.3) is satisfied, so a T-universal regulator exists. Such a regulator is obtained by, for example, setting $\rho_0 = 1$ and $R = 0$ in (7.4). If $\alpha_1 = 2$ and $\alpha_2 = 1$ and the initial conditions are $y_0 = y_1 = 1$, this yields the error depicted in Figure 6.1. The dashed line in the same figure is the tracking error obtained by setting $u_t \equiv 0$.

Next, let us take $c = 0.75$, while a and b remain the same. Then ψ becomes unstable, so a T-universal regulator fails to exist. Although condition (7.3) fails, we could still obtain asymptotic tracking by using a universal tracking regulator, constructed as in Theorem 4.1, provided condition (7.6) holds, and we shall present a simulation for this case in the end of the section.

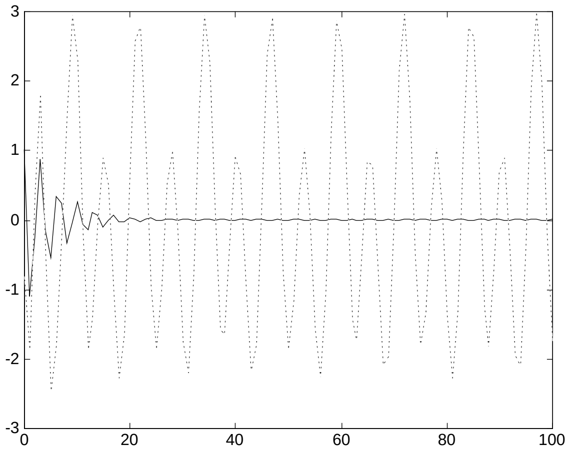

Figure 6.1

We now add an harmonic disturbance

$$w_t = \alpha_3 \cos(\theta_3 t + \varphi_3) + \alpha_4 \cos(\theta_4 t + \varphi_4) \tag{7.8}$$

in the system (7.1), where θ_3, θ_4 are given, but α_3, α_4 and φ_3, φ_4 are unknown. Suppose we want determine a optimal universal regulator for the cost function

$$\Phi = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \{|z_t - r_t|^2 + \beta u_t^2\}, \quad \beta \geq 0. \tag{7.9}$$

Since the matrices Q, S and R in (5.4) become

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} c \\ 0 \end{bmatrix}, \quad R = \beta + c^2,$$

a simple calculation yields

$$\Pi(\lambda) = \left| \frac{\psi(\lambda)}{\chi(\lambda)} \right|^2 + \beta$$

for (5.8), and therefore the strong frequency domain condition (5.9) is always satisfied if $\beta > 0$, so any optimal universal regulator (7.2) is optimal in the larger class $\mathcal{N}$ of possibly nonlinear regulators described in Section 6. If $\beta = 0$, the strong frequency domain condition will fail if and only if the polynomial ψ has a root on the unit circle, while the weak frequency condition (5.10) will still hold provided we avoid choosing any of the frequencies in (7.5) and (7.8) so that $e^{i\theta_1}$, $e^{i\theta_2}$, $e^{i\theta_3}$ or $e^{i\theta_4}$ is such a root.

Next, let us consider the interpolation condition (5.12). Clearly, $F(\lambda)$ defined by (4.3), is identically one, and a straight-forward calculation yields

$$U(\lambda) = -\frac{1 + c\lambda\chi(\lambda^{-1})}{|\psi(\lambda)|^2 + \beta|\chi(\lambda)|^2}, \quad \hat{U}(\lambda) = \frac{\psi(\lambda^{-1})\chi(\lambda)}{|\psi(\lambda)|^2 + \beta|\chi(\lambda)|^2}$$

for any λ on the unit circle. In order to construct an optimal universal regulator we need to choose a stable polynomial

$$\rho(\lambda) = \lambda^5 + \rho_1\lambda^4 + \rho_2\lambda^3 + \rho_3\lambda^2 + \rho_4\lambda + \rho_5,$$

of degree at least five. The parameters $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ as well as β will be available for tuning in order to improve the overall design. Then, defining the real numbers $u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4$ via

$$\begin{aligned} \rho(e^{i\theta_j})U(e^{i\theta_j}) &= u_j + iv_j \quad \text{for } j = 1, 2 \\ \rho(e^{i\theta_j})\hat{U}(e^{i\theta_j}) &= u_j + iv_j \quad \text{for } j = 3, 4, \end{aligned}$$

it is easily seen that the polynomials

$$\begin{aligned} R(\lambda) &= R_1\lambda^3 + R_2\lambda^2 + R_3\lambda + R_4 \\ L(\lambda) &= L_1\lambda^3 + L_2\lambda^2 + L_3\lambda + L_4 \end{aligned}$$

will satisfy the interpolation conditions (5.12a) if and only if its coefficients satisfy the linear system of equations

$$\begin{bmatrix} \cos 3\theta_1 & \cos 2\theta_1 & \cos\theta_1 & 1 \\ \sin 3\theta_1 & \sin 2\theta_1 & \sin\theta_1 & 0 \\ \cos 3\theta_2 & \cos 2\theta_2 & \cos\theta_2 & 1 \\ \sin 3\theta_2 & \sin 2\theta_2 & \sin\theta_2 & 0 \end{bmatrix} \begin{bmatrix} R_1 & L_1 \\ R_2 & L_2 \\ R_3 & L_3 \\ R_4 & L_4 \end{bmatrix} = \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \\ u_2 & u_4 \\ v_2 & v_4 \end{bmatrix}.$$

Consequently, by Theorem 5.1, (7.7) is an optimal universal regulator if $R(\lambda)$ and $L(\lambda)$ are determined in this way.

For an example, take as before $a = 0.4$, $b = 0.7$, and $c = 0.75$. Moreover, we choose a disturbance (7.8) with frequencies $\theta_3 = 0.5$ and $\theta_4 = 0.3$, while the harmonic reference signal (7.5) has the same frequencies $\theta_1 = 1.0$, $\theta_2 = 0.5$ as in the first simulation. In Figure 6.2 we illustrate the tracking error of the optimal universal regulator corresponding to a polynomial ρ with roots $0.3 \pm 0.3i$, $0.3 \pm 0.2i$, 0.5 and $\beta = 0.75$. The amplitudes in (7.5) and (7.8) have been taken to be $\alpha_1 = 2$, $\alpha_2 = \alpha_3 = 1$ and $\alpha_4 = 4$, and the initial conditions are $y_0 = y_1 = 1$. As before, the dashed line is the tracking error obtained by setting $u_t \equiv 0$. Remember that, since $\beta \neq 0$, the control energy is also damped, so there is a certain trade off here. We remark that it is important to tune the free parameters to obtain good properties of the regulator. In particular, the transients, which do not affect the cost function, can change dramatically with different choices of free parameters.

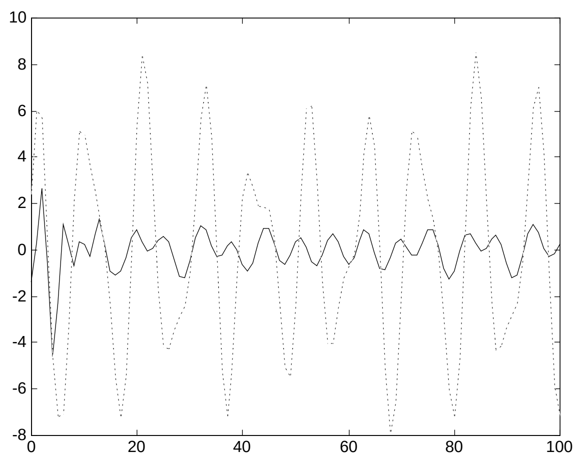

Figure 6.2

Now, setting $c = 0$ and $\alpha_3 = \alpha_4 = \beta = 0$ instead, while keeping all the other parameters the same, we obtain the errors in Figure 6.3. As seen, the error goes asymptotically to zero, despite the fact that condition (7.3) is not fulfilled so that a T-universal regulator does not exist. In fact, by Theorem 4.1, this is a universal tracking regulator, which exists since $\psi(\lambda) = \lambda \neq 0$ on the unit circle. In order to speed up the convergence, the roots of ρ have been reset at $0.7 \pm 0.1i$, $0.3 \pm 0.2i$ and 0.8. Since now we do not have the disturbance frequencies $\theta_3 = 0.5$ and $\theta_4 = 0.3$, we could choose another $R(\lambda)$ to possibly get a universal tracking regulator with a better transient.

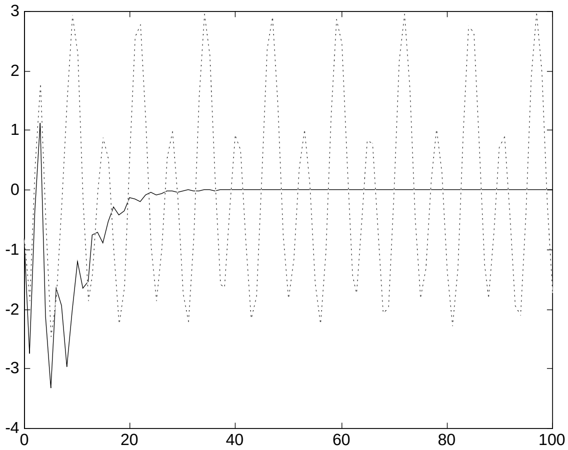

Figure 6.3

8. Conclusions

In this paper we have given *complete* characterizations of regulators which satisfy certain tracking specifications and which are universal in the sense that they are independent of disturbances and tracking signals and apply regardless of the values of these.

As a preliminary, we considered a problem of asymptotic tracking of an arbitrary signal r_t, and we characterized all regulators which are universal with respect to the choice of r_t. We showed that such universal regulators exist only under very special conditions. These condition can be considerably relaxed if the reference signal is exchanged for a harmonic signal with known frequencies but unknown amplitudes and phases, and we want the regulator to be universal in the sense that it achieves asymptotic tracking for all choices of amplitudes and phases. Then, if the dimension μ of the reference signal is no larger than the dimension k of the control, such a regulator exist under mild conditions. This is in harmony with other results in the literature [4, 5, 6, 7, 8, 13, 16], where, however, the continuous-time case is considered. We provided complete solutions of these problems in discrete time, and our proof is considerably simpler.

If the system is also corrupted by a harmonic disturbance w_t, asymptotic tracking may still be possible provided the dimension ℓ of the disturbance is no larger than the dimension m of the output available for feedback. However, if a certain rank condition fails, which in particular is the case if $\mu > k$, asymptotic tracking is not possible, but a steady state error will remain. Therefore, we considered next an optimal control problem to damp the steady-state tracking error, also giving the option to damp internal system variables. We characterized the class of all optimal regulators which are universal in the sense that they are optimal for all choices of the amplitudes of r_t and w_t. Such regulators were shown to exist if the weak frequency domain condition holds and $\ell \leq m$. On the other hand, if $m < \ell$, there are always algebraic conditions on the system parameters, implying that universality is not a generic property in this case.

We have also shown that all optimal universal regulators can be chosen as linear even if the optimization is over a very large class of nonlinear regulators, provided the strong frequency domain condition holds. We have given complete characterizations of all linear optimal universal regulators in terms of parameterizations containing many free parameters. This allows for a considerable amount of design freedom, which can be used to satisfy other design specifications via loop shaping. Indeed, we stress that our solutions are optimal in the sense stated in this paper only, and that other desirable design specifications may not be satisfied for an arbitrary universal optimal regulator.

Appendix A. Proof of Lemma 5.5

Since x_t and u_t tend exponentially to the harmonic components (4.7), only these contribute to the cost function (5.3), and consequently the usual limit (rather than just limsup) does exist in (5.2), and it is given by $\Phi = \sum_{j=1}^N \Phi^{(j)}$ where

$$\Phi^{(j)} = \Lambda(x^{(j)}, u^{(j)}) - r^{(j)*}(Hx^{(j)} + Ju^{(j)}) - (Hx^{(j)} + Ju^{(j)})^* r^{(j)} + r^{(j)*} r^{(j)} \tag{A.1}$$

for $j = 1, 2, \dots, N$. In fact, this follows from the argument leading to (4.11). Now, in view of the constraint (1.1a),

$$x^{(j)} = A_{\lambda_j}(Bu^{(j)} + Ew^{(j)}), \tag{A.2}$$

and therefore (A.1) takes the form

$$\Phi^{(j)} = u^{(j)*}\Pi(\lambda_j)u^{(j)} + p_j^* u^{(j)} + u^{(j)*}p_j + q_j, \tag{A.3}$$

where $\Pi(\lambda_j) > 0$ if the weak frequency domain condition (5.10) is fulfilled. Here q_j is given by (5.18), and

$$p_j = [QA_{\lambda_j}B + S]^* A_{\lambda_j} Ew^{(j)} - [HA_\lambda B + J]^* r^{(j)}. \tag{A.4}$$

Therefore, assuming that the weak frequency domain condition (5.10) holds so that $\Pi(\lambda_j) > 0$ for $j = 1, 2, \dots, N$, we may complete squares in (A.3) to obtain

$$\Phi^{(j)} = (u^{(j)} - u_{\mathrm{opt}}^{(j)})^* \Pi(\lambda_j)(u^{(j)} - u_{\mathrm{opt}}^{(j)}) + \Phi_{\min}^{(j)}, \tag{A.5}$$

where

$$u_{\mathrm{opt}}^{(j)} = -\Pi(\lambda_j)^{-1} p_j, \quad \Phi_{\min}^{(j)} = q_j - {p_j}^* \Pi(\lambda_j)^{-1} p_j. \tag{A.6}$$

From this the equations of the lemma follow readily.

Appendix B. Proof of Lemma 6.3

The proof is similar, *mutatis mutandis*, to the one given in [21, Section II]. Recall from (5.2) that the cost function can be written

$$\Phi = \limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \Omega(x_t, u_t, t), \tag{B.1}$$

where

$$\Omega(x, u, t) = \Lambda(x, u) - r_t^* Hx - x^* H^* r_t - r_t^* Ju - u^* J^* r_t + {r_t}^* r_t \tag{B.2}$$

with $\Lambda(x, u)$ being the quadratic form (5.3). Next, introduce the Lyapunov function

$$V(x, t) = x^* Px + {p_t}^* x + x^* p_t + q_t, \tag{B.3}$$

where P is the unique stabilizing solution of (6.4), $\{p_t\}_{t\in\mathbb{Z}_+}$ is given by (6.8) and $\{q_t\}_{t\in\mathbb{Z}_+}$ satisfies (6.12). Then, along the trajectory of (6.3),

$$V(x_{t+1}, t+1) - V(x_t, t) + \Omega(x_t, u_t, t) = (u_t - Kx_t - \pi_t)^* \hat{R}(u_t - Kx_t - \pi_t) \tag{B.4}$$

where π_t is given by (6.7).

In fact, inserting (6.3) and completing squares in the left member of (B.4) yields the right member of (B.4) plus a number of terms which are either quadratic in x_t, linear in x_t, or constant with respect to x_t. The quadratic terms cancel due to the fact that P satisfies the algebraic Riccati equation (6.4), and the constant terms cancel due to (6.12). Finally the linear terms cancel provided

$$p_t = \Gamma^* p_{t+1} + \Gamma^* P v_t - (H + JK)^* r_t,$$

which has the unique bounded solution (6.8), since Γ is a stable matrix.

Now, set $V_t := V(x_t, t)$ and $\Omega_t := \Omega(x_t, u_t, t)$, where (x_t, u_t) is an admissible process, and sum (B.4) from $t = 0$ to $t = T$ to obtain

$$\frac{1}{T}(V_{T+1} - V_0) + \frac{1}{T}\sum_{t=0}^{T} \Omega_t = \frac{1}{T}\sum_{t=0}^{T} (u_t - Kx_t - \pi_t)^* \hat{R}(u_t - Kx_t - \pi_t).$$

By virtue of condition (6.2) and the boundedness of p_t,

$$\frac{1}{T}(V_{T+1} - V_0) = \frac{1}{T} q_{T+1} + o(1),$$

where of course the last term tends to zero as $T \to \infty$. Consequently, for any admissible (x_t, u_t), the cost function (B.1) becomes

$$\Phi = \limsup_{T\to\infty} \{\frac{1}{T}\sum_{t=0}^{T} (u_t - Kx_t - \pi_t)^* \hat{R}(u_t - Kx_t - \pi_t) - \frac{1}{T} q_{T+1}\}, \tag{B.5}$$

and therefore, since $\hat{R} > 0$,

$$\Phi \geq \limsup_{T\to\infty} \frac{1}{T}(-q_{T+1}) \tag{B.6}$$

for any admissible control. Clearly, equality would be achieved if we could take (x_t, u_t) to satisfy (6.9) since ϵ_t does not contribute to Φ by virtue of (6.10). Hence it remains to prove that such a process satisfies the stability condition (6.2). To this end, insert (6.9) in (6.3) to obtain

$$x_{t+1} = \Gamma x_t + B(\pi_t + \epsilon_t) + v_t. \tag{B.7}$$

Since $\{\pi_t\}_{t\in\mathbb{Z}_+}$ and $\{v_t\}_{t\in\mathbb{Z}_+}$ are bounded, $\{\epsilon_t\}_{t\in\mathbb{Z}_+}$ satisfies (6.10) and Γ is a stability matrix, $\{x_t\}_{t\in\mathbb{Z}_+}$ satisfies the weak stability condition (6.2). The last statement follows immediately from (B.5) and (B.6).

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