Distrbutionally Robust Optimization: A Marriage of Robust Optimization and Stochastic Programming

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Agenda

- Modeling Uncertainty
- DRO
- DRO with Recourse
- Conclusions

References for this talk

- Chen, W. and M. Sim (2008): Goal Driven Optimization, forthcoming in Operations Research
- Chen, W., M. Sim, J. Sun and CP Teo (2008): From CVaR to Uncertainty Sets: Implications in Joint Chance Constrained Optimization, forthcoming in Operations Research
- Chen, Xin, M. Sim and P. Sun (2007): A Robust Optimization Perspective of Stochastic Programming, Operations Research, 344-35755(6), 1058-1071
- Chen, Xin, M. Sim, P. Sun, and J. Zhang (2008): A Linear Decision based Approximation Approach to Stochastic Programming, Operations Research, 56(2), 344-357

References for this talk

- K. Natarajan, M. Sim and J. Uichanco (2008): Tractable Robust Expected Utility and Risk Models for Portfolio Optimization, forthcoming Mathematical Finance
- K. Natarajan, D. Pachamanova , M. Sim (2008): Incorporating Asymmetric Distributional Information in Robust Value-at-Risk Optimization, Management Science, 54(3), 573-585
- See, CT and M. Sim (2008): Robust Approximation to Multi-Period Inventory Management, under 3rd revision in Operations Research

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- Modeling Uncertainty
- Robust Linear Optimization
- Robust Linear Optimization with Recourse
- Conclusions

Modeling Uncertainty: Probability Distributions

- Data represented as random variables with known distributions
  - Stochastic/Dynamic Programming approach
  - Information required
    - Sample space (all possible outcomes, usually exponential or infinite)
    - Distributions (probability of outcome)
  - Advantages
    - A widely accepted method in math/statistics
    - Able to quantify expectations such as evaluating probability of outcomes

Modeling Uncertainty: Probability Distributions

- Decision based on taking expectations

$$\mathsf{E}(u(x, \tilde{z}))$$

Modeling Uncertainty: Probability Distributions

- Disadvantages
  - Quantifying expectation is computationally intractable

NP-hard to evaluate accurately

$$\mathrm{P}(\tilde{\boldsymbol{r}}'\boldsymbol{x} > 1)$$

where $\tilde{\boldsymbol{r}}$ is iid uniformly distributed in $[-1, 1]$.

- Shown by Nemirovski and Shapiro 2004, based on a result of Khachiyan in computing volume of polytope

Modeling Uncertainty: Probability Distributions

- Practically prohibitive to obtained exact distributions
  - Absence or limited historical data
  - Reliability of historical data in predicting outcomes; non-stationary
  - Difficulty of describing multivariate random variable
- How about using empirical distributions or data driven approaches?

Modeling Uncertainty: Probability Distributions

- A Portfolio Optimization Case Study
  - 24 small cap stocks from different industry categories
  - Historical returns from April 17 1998 to June 1, 2006
  - Return and Covariance estimated from initial 80% of the data. Evaluate performance on last 20%.

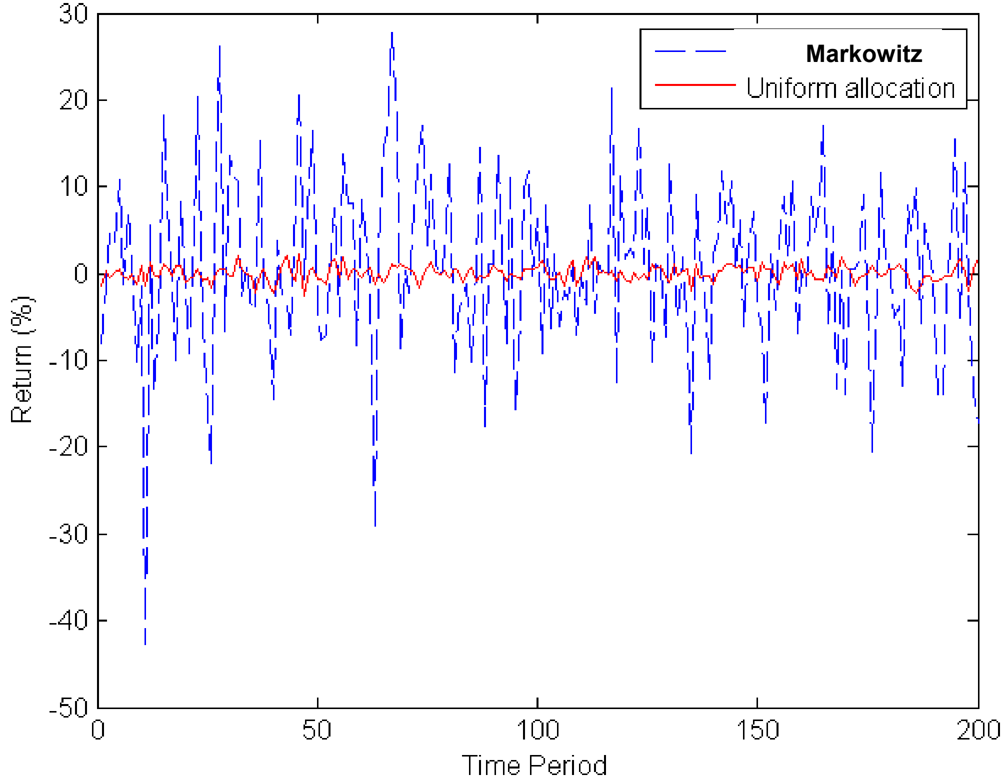
Modeling Uncertainty: Probability Distributions

- Markowitz model
  - $\tilde{\boldsymbol{r}}$: Vector of stock returns. Mean $\boldsymbol{\mu}$, Co-varience Σ. (Estimated)

$$
\begin{aligned}
\text{min} \quad & \boldsymbol{x}'\Sigma\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{\mu}'\boldsymbol{x} = \boldsymbol{\mu}'\mathbf{1}/n \\
& \boldsymbol{x}'\mathbf{1} = 1,
\end{aligned}
$$

Modeling Uncertainty: Probability Distributions

- Optimizing over unreliable historical data can be catastrophic!!!



Modeling Uncertainty: Uncertainty Set

- Data represented as uncertainty set
  - Robust Optimization approach
  - Information required
    - Convex hull of data realization described in tractable forms:
      - Polyhedral
      - Conic quadratic, etc

Modeling Uncertainty: Uncertainty Set

- Decision based on worst-case value over the uncertainty set

$$\min_{z \in \mathcal{W}} u(x, z)$$

Modeling Uncertainty: Uncertainty Set

- Ellsberg Paradox
  - Box 1: 50 red balls and 50 blue balls
  - Box 2: 100 red and blue balls with unknown proportions

Payoffs: $1,000,000 for choosing a red ball.
Which box will you choose?

Modeling Uncertainty: Uncertainty Set

- Ellsberg Paradox
  - Box 1: 50 red balls and 50 blue balls
  - Box 2: 100 red and blue balls with unknown proportions

Payoffs: $1,000,000 for choosing a blue ball.
Which box will you choose?

Decision Maker is Ambiguity Averse!!

Modeling Uncertainty: Uncertainty Set

- Advantages
  - Less information is required
    - Convex hull versus Sample space
    - Distribution free
  - Computational tractable for many important classes of optimization.
  - Quantifiable approximation exists for some hard ones
  - Natural way of describing uncertainty in certain applications
    - Engineering applications

Modeling Uncertainty: Uncertainty Set

- Disadvantages
  - Unable to evaluate expectations including probability measure
  - How do we choose the right uncertainty set?
    - Requires domain knowledge

Modeling Uncertainty: Descriptive Statistics

- Data represented as random variable over a family of distributions characterized by its descriptive statistics
  - Distributionally Robust Optimization approach
  - Information required
    - Convex hull of support
    - Descriptive statistics: means, standard deviations, directional deviations, independence etc.

Modeling Uncertainty: Descriptive Statistics

- Advantages
  - Descriptive statistics can be derived from data
  - Solutions are robust to distributional assumptions
  - Moderate information needed
  - Tractable approximations available
  - Compute bounds on expectations
  - Far less conservative than worst case

Modeling Uncertainty: Descriptive Statistics

- Decision based on worst-case value expectation over the family of distributions

$$\inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}(u(\boldsymbol{x},\tilde{\boldsymbol{z}}))$$

$\mathbb{F}$: Family of distibutions that contains the true distribution.

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Distributionally Robust Optimization

- A typical linear optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \tilde{\boldsymbol{a}}_i'\boldsymbol{x} \leq \tilde{b}_i, \quad i \in \{1, \ldots, m\} = \mathcal{M} \\ & \boldsymbol{x} \in \Re^n, \end{array}$$

$\tilde{\boldsymbol{a}}_i, \tilde{b}_i$: Potentially uncertain data

Distributionally Robust Optimization

- WLOG, we assume data is affinely dependent on a set of N primitive uncertainties: $\tilde{\boldsymbol{z}} \triangleq (\tilde{z}_1, \ldots, \tilde{z}_N)$.
  - Provision for linear correlations among data

$$\tilde{\boldsymbol{a}}_i = \boldsymbol{a}_i(\tilde{\boldsymbol{z}}) \triangleq \boldsymbol{a}_i^0 + \sum_{j=1}^{N} \boldsymbol{a}_i^j \tilde{z}_j$$

$$\tilde{b}_i = b_i(\tilde{\boldsymbol{z}}) \triangleq b_i^0 + \sum_{j=1}^{N} b_i^j \tilde{z}_j$$

Distributionally Robust Optimization

For notational convenience, ignore subscript i

$$\boldsymbol{a}(\tilde{\boldsymbol{z}})'\boldsymbol{x} - b(\tilde{\boldsymbol{z}}) = \underbrace{\boldsymbol{a}^{0\prime}\boldsymbol{x} - b^0}_{=y^0} + \sum_{j=1}^{N} \underbrace{(\boldsymbol{a}^{j\prime}\boldsymbol{x} - b^j)}_{=y_j} \tilde{z}_j$$

$$\boldsymbol{a}(\tilde{\boldsymbol{z}})'\boldsymbol{x} \le b(\tilde{\boldsymbol{z}})$$

$$\Updownarrow$$

$$y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} \le 0$$

Distributionally Robust Optimization

- Robust Optimization approach
  - Find solutions that remains feasible for all data in uncertainty sets, a.k.a **Robust Counterpart**

$$y^0 + \boldsymbol{y}'\boldsymbol{z} \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{U}$$

Distributionally Robust Optimization

- Stochastic Programming approach
  - Chance constraint (Charnce, Cooper and Symonds 58)

$$\mathbb{P}(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} \leq 0) \geq 1 - \epsilon$$

$\tilde{\boldsymbol{z}}$: A multivariate random variable with known distribribution.

  - Typically intractable problem

Distributionally Robust Optimization

Assume $\tilde{z}$ is a multivariate random variable with distribution $\mathbb{P}$ from a family of distributions $\mathbb{F}$.

For all $\mathbb{P} \in \mathbb{F}$, the random variable $\tilde{z}$ has the same descriptive statistics such as same mean, support, covariance, deviation measures, and so forth.

Distributionally Robust Optimization

- Robust chance constrained problem

$$\inf_{\mathbb{P}\in\mathbb{F}} \mathbb{P}(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} \le 0) \ge 1 - \epsilon$$

  - Generally intractable
  - Tractable formulation for family of distribution with infinite support, known mean and covariance, (Bertsimas and Pospescu, 2004, El Ghaoui et al, 2003)

Distributionally Robust Optimization

- Should we choose an uncertainty set large enough to contain most of the samples?

Choose $\mathcal{U}$ such that

$$\mathbb{P}(\tilde{\boldsymbol{z}} \in \mathcal{U}) \geq 1 - \epsilon$$

so that

$$y^0 + \boldsymbol{y}'\boldsymbol{z} \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{U}$$
$$\Downarrow$$
$$\mathbb{P}(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} \leq 0) \geq 1 - \epsilon$$

Distributionally Robust Optimization

Suppose $\tilde{z}_j$ iid two point symmetrically distributed taking values in $\{-1, 1\}$. Choose ellipsoidal uncertainty set

$$\mathcal{E}_r = \{\boldsymbol{z} \ : \ \|\boldsymbol{z}\|_2 \leq r\}$$

It is well-known that

$$y_0 + \boldsymbol{y}'\boldsymbol{z} \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{E}_4$$
$$\Downarrow$$
$$\mathbb{P}(y_0 + \boldsymbol{y}'\boldsymbol{z} \leq 0) \geq 1 - \exp(-4^2/2) \approx 0.99966$$

Distributionally Robust Optimization

$$y_0 + \boldsymbol{y}'\boldsymbol{z} \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{E}_4$$

$$\Downarrow$$

$$\mathbb{P}(y_0 + \boldsymbol{y}'\boldsymbol{z} \leq 0) \geq 0.99966$$

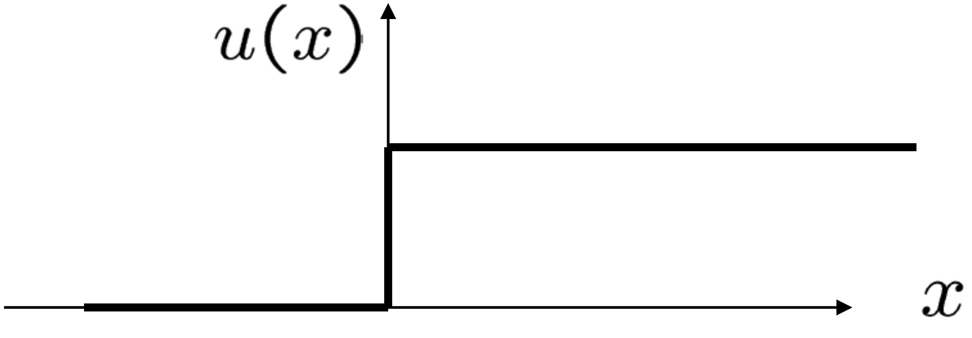
However, if dimension of $\boldsymbol{z}$ is greater than 16,

$$\mathbb{P}(\tilde{\boldsymbol{z}} \in \mathcal{E}_4) = 0!!!$$

Distributionally Robust Optimization

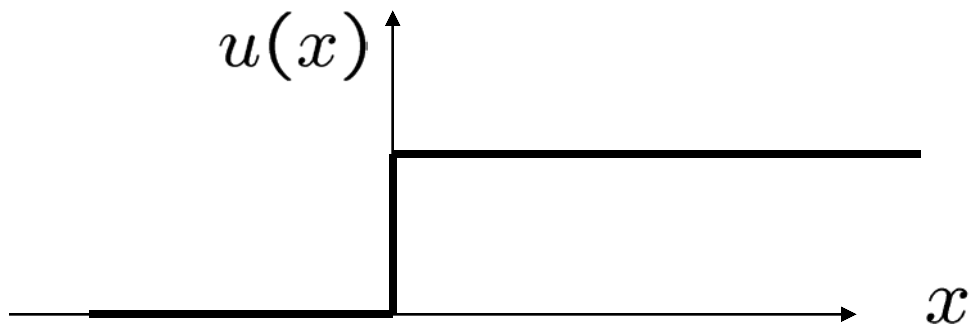
- Key Idea: Convexification of chance constrained problem.

$$\mathbb{P}(y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} \leq 0) \geq 1 - \epsilon$$
$$\Updownarrow$$
$$\mathbb{E}(u(-y_0 - \boldsymbol{y}'\tilde{\boldsymbol{z}})) \geq 1 - \epsilon$$

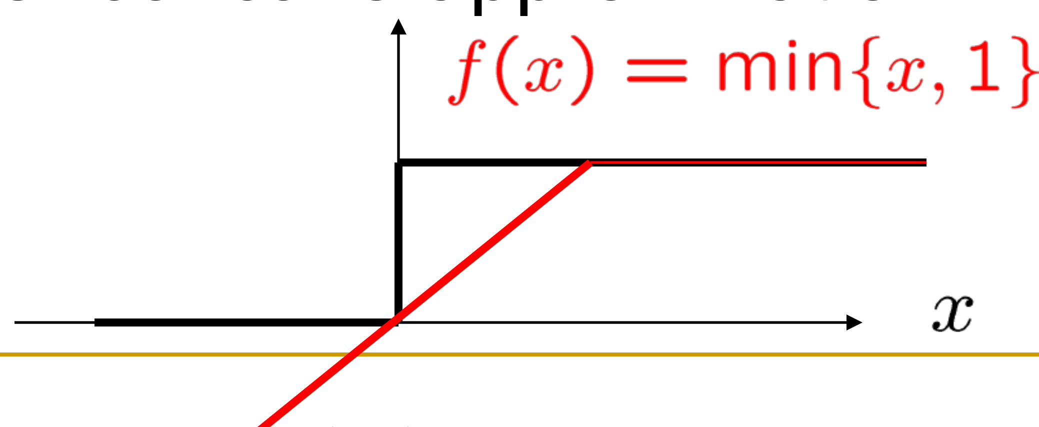


Distributionally Robust Optimization

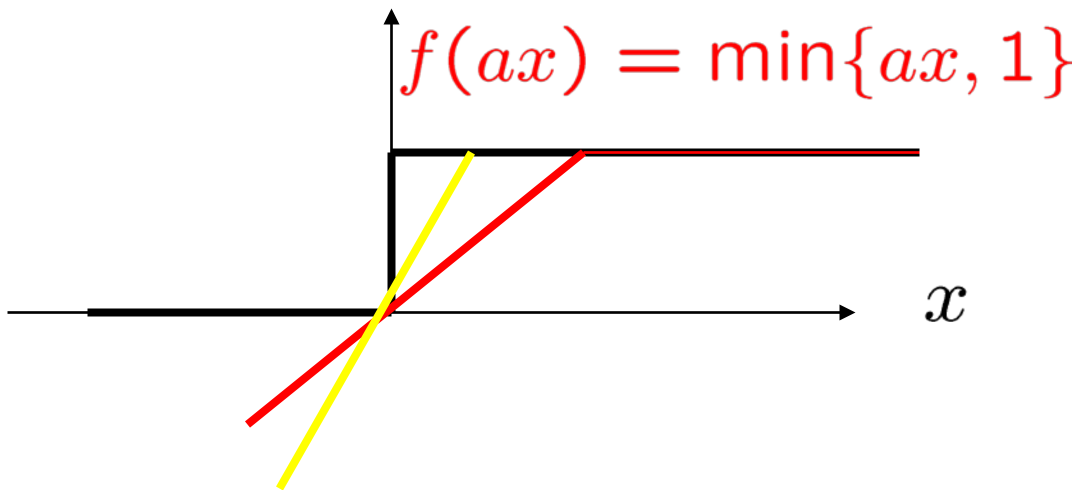
- Step utility function is not concave!!



- Consider concave approximation



Distributionally Robust Optimization



$$\exists a > 0 \;:\; \mathbb{E}(\min\{-ay_0 - a\boldsymbol{y}'\tilde{\boldsymbol{z}}, 1\}) \geq 1 - \epsilon$$
$$\Downarrow$$
$$\mathbb{E}(u(-y_0 - \boldsymbol{y}'\tilde{\boldsymbol{z}})) \geq 1 - \epsilon$$
$$\Updownarrow$$
$$\mathbb{P}(y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} \leq 0) \geq 1 - \epsilon$$

Distributionally Robust Optimization

$$\exists a > 0 \;\; : \;\; \mathbb{E}(\min\{-ay_0 - a\boldsymbol{y}'\tilde{\boldsymbol{z}}, 1\}) \geq 1 - \epsilon$$

$$\Updownarrow$$

$$\mathbb{E}(\min\{-ay_0 - a\boldsymbol{y}'\tilde{\boldsymbol{z}} - 1, 0\}) + 1 \geq 1 - \epsilon$$

$$\Updownarrow$$

$$-\mathbb{E}((ay_0 + a\boldsymbol{y}'\tilde{\boldsymbol{z}} - 1)^+)/\epsilon \geq -1$$

$$\Updownarrow$$

$$-\mathbb{E}((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + 1/a)^+)/\epsilon \geq -1/a$$

$$\cdots$$

Distributionally Robust Optimization

$$-\mathbb{E}((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + 1/a)^+)/\epsilon \geq -1/a$$

$$\Updownarrow$$

$$\exists \beta \geq 0 \quad : \quad -\mathbb{E}((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \beta)^+)/\epsilon \geq -\beta$$

$$\Updownarrow$$

$$\beta + \mathbb{E}((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \beta)^+)/\epsilon \leq 0$$

$$\Updownarrow$$

$$\inf_\beta \left\{-\beta + \mathbb{E}((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \beta)^+)/\epsilon\right\} \leq 0$$

Distributionally Robust Optimization

- Conditional Value-at-Risk (CVaR)
  - Popularized by Rockafellar and Uryasev
  - Best possible convex approximation of chance constrained problems. (Foellmer and Scheid 2004, Nemirovski and Shapiro 2006)

$$\rho_{1-\epsilon}(\tilde{r}) \triangleq \inf_{\beta} \left\{ \beta + \mathbb{E}((\tilde{r} - \beta)^{+})/\epsilon \right\}$$

where $\tilde{r}$ is a random variable.

Distributionally Robust Optimization

$$\rho_{1-\epsilon}(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}) \leq 0$$

$$\Downarrow$$

$$\mathbb{P}(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} \leq 0) \geq 1 - \epsilon$$

Distributionally Robust Optimization

$$\rho_{\mathbb{P},1-\epsilon}(y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}) = \inf_{\beta}\left\{\beta + \frac{1}{\epsilon}\mathbb{E}_{\mathbb{P}}((y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}-\beta)^+)\right\}$$

Convex in $(y^0, \boldsymbol{y})$... but how to compute

$$\mathbb{E}_{\mathbb{P}}((y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}-\beta)^+)?$$

How about using only descriptive statistics?

$$\rho_{\mathbb{F},1-\epsilon}(y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}) = \inf_{\beta}\left\{\beta + \frac{1}{\epsilon}\sup_{\mathbb{P}\in\mathbb{F}}\mathbb{E}_{\mathbb{P}}((y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}-\beta)^+)\right\}$$

Distributionally Robust Optimization

- Upper bounds on E(.)$^+$

$$\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}((y^0 + \tilde{\boldsymbol{z}}'\boldsymbol{y})^+) \leq \pi(y^0, \boldsymbol{y})$$

$$\rho_{\mathbb{F},1-\epsilon}(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}) \leq \eta_{1-\epsilon}(y^0, \boldsymbol{y})$$

where

$$\eta_{1-\epsilon}(y^0, \boldsymbol{y}) \triangleq \inf_{\beta} \left\{\beta + \pi(y^0 - \beta, \boldsymbol{y})/\epsilon\right\}$$

Distributionally Robust Optimization

- Approximation idea:

$$\eta_{1-\epsilon}(y^0, \boldsymbol{y}) \leq 0$$

$$\Downarrow$$

$$\rho_{\mathbb{F},1-\epsilon}(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}) \leq 0$$

$$\Downarrow$$

$$\inf_{\mathbb{P}\in\mathbb{F}} \mathbb{P}(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} \leq 0) \geq 1 - \epsilon$$

Distributionally Robust Optimization

Suppose $\pi(y^0, \boldsymbol{y})$ is convex, positive homogenous, and satisfies $\pi(a, \mathbf{0}) = a^+$, then

$$\eta_{1-\epsilon}(y^0, \boldsymbol{y}) \leq 0$$

is equivalent to the robust counterpart

$$y^0 + \boldsymbol{y}'\boldsymbol{z} \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{U}(\epsilon)$$

for some uncertainty set $\mathcal{U}(\epsilon)$.

-Chen, Sim, Sun and Teo (2007)

Distributionally Robust Optimization

Implications: Finding an uncertainty set is the same as finding a convex, positive homogenous function, $\pi(y_0, \boldsymbol{y})$ that is an upper bound of $\mathbb{E}_{\mathbb{P}}((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+)$ for all $\mathbb{P} \in \mathbb{F}$.

Distributionally Robust Optimization

Model of uncertainty: Assume primitive uncertainty $\tilde{\boldsymbol{z}}$ is a multivariate random variable with convex hull of sample space given by $\mathcal{W} = \{\boldsymbol{z} : -\underline{\boldsymbol{z}} \leq \boldsymbol{z} \leq \bar{\boldsymbol{z}}\}$. Let $\mathbb{F}$ be a family of distributions such that for all $\mathbb{P} \in \mathbb{F}$:

1. $\mathbb{E}_{\mathbb{P}}(\tilde{\boldsymbol{z}}) = \mathbf{0}$
2. $\mathbb{E}_{\mathbb{P}}(\tilde{\boldsymbol{z}}\tilde{\boldsymbol{z}}') = \boldsymbol{\Sigma}$, which is positive definite.

Distributionally Robust Optimization

3. A subset $\mathcal{I} \subseteq \{1, \ldots, N\}$, such that $\tilde{z}_j$, $j \in \mathcal{I}$ are stochastically independent.

$$p_j = \begin{cases} \sigma_f(\mathbb{F}, \tilde{z}_j) & \text{if } j \in \mathcal{I} \\ \infty & \text{otherwise} \end{cases}$$

$$q_j = \begin{cases} \sigma_b(\mathbb{F}, \tilde{z}_j) & \text{if } j \in \mathcal{I} \\ \infty & \text{otherwise} \end{cases}$$

Distributionally Robust Optimization

- X. Chen, S. and P. Sun, 2006

Suppose $\mathbb{E}_{\mathbb{P}}(\tilde{z}) = 0$ for all $\mathbb{P} \in \mathbb{F}$:

Forward Deviation:

$$\sigma_f(\mathbb{F}, \tilde{z}) = \sup_{\theta > 0, \mathbb{P} \in \mathbb{F}} \left\{ \sqrt{2 \frac{\ln(\mathbb{E}_{\mathbb{P}}(\exp(\theta \tilde{z})))}{\theta^2}} \right\}.$$

Backward Deviation:

$$\sigma_b(\mathbb{F}, \tilde{z}) = \sup_{\theta > 0, \mathbb{P} \in \mathbb{F}} \left\{ \sqrt{2 \frac{\ln(\mathbb{E}_{\mathbb{P}}(\exp(-\theta \tilde{z})))}{\theta^2}} \right\}.$$

Distributionally Robust Optimization

- Worst case deviations with given support

For a family of distributions $\mathbb{F}$ such that $\tilde{z}$ has zero mean and support $[-\underline{z}, \bar{z}]$, $\underline{z}, \bar{z} > 0$, then

$$\sigma_f(\mathbb{F}, \tilde{z}) = \frac{\underline{z} + \bar{z}}{2}\sqrt{g\left(\frac{\underline{z} - \bar{z}}{\underline{z} + \bar{z}}\right)}$$

and

$$\sigma_b(\mathbb{F}, \tilde{z}) = \frac{\underline{z} + \bar{z}}{2}\sqrt{g\left(\frac{\bar{z} - \underline{z}}{\underline{z} + \bar{z}}\right)},$$

where

$$g(\mu) = 2\max_{s>0}\frac{\phi_\mu(s) - \mu s}{s^2},$$

and

$$\phi_\mu(s) = \ln\left(\frac{e^s + e^{-s}}{2} + \frac{e^s - e^{-s}}{2}\mu\right).$$

Distributionally Robust Optimization

- Unified bound on E(.)$^+$

$$
\begin{array}{rl}
\pi(y_0, \boldsymbol{y}) = \min & r_1 + r_2 + r_3 + r_4 + r_5 \\
\text{s.t.} & y_{10} + \sum_{j:\bar{z}_j<\infty} s_j \bar{z}_j + \sum_{j:\underline{z}_j<\infty} t_j \underline{z}_j \le r_1 \\
& s_j = 0 \ \forall j : \bar{z}_j = \infty, \qquad t_j = 0 \ \forall j : \underline{z}_j = \infty \\
& 0 \le r_1 \\
& \boldsymbol{s} - \boldsymbol{t} = \boldsymbol{y}_1 \\
& \boldsymbol{s}, \boldsymbol{t} \ge 0 \\
& \sum_{j:\bar{z}_j<\infty} d_j \bar{z}_j + \sum_{j:\underline{z}_j<\infty} h_j \underline{z}_j \le r_2 \\
& d_j = 0 \ \forall j : \bar{z}_j = \infty, \qquad h_j = 0 \ \forall j : \underline{z}_j = \infty \\
& y_{20} \le r_2 \\
& \boldsymbol{d} - \boldsymbol{h} = -\boldsymbol{y}_2 \\
& \boldsymbol{d}, \boldsymbol{h} \ge 0 \\
& \frac{1}{2} y_{30} + \frac{1}{2} \|(y_{30}, \boldsymbol{\Sigma}^{1/2} \boldsymbol{y}_3)\|_2 \le r_3 \\
& \inf_{\mu>0} \frac{\mu}{e} \exp\left(\frac{y_{40}}{\mu} + \frac{\|\boldsymbol{u}\|_2^2}{2\mu^2} \right) \le r_4 \\
& u_j \ge p_j y_{4j} \ \forall j : p_j < \infty, \qquad y_{4j} \le 0 \ \forall j : p_j = \infty \\
& u_j \ge -q_j y_{4j} \ \forall j : q_j < \infty, \qquad y_{4j} \ge 0 \ \forall j : q_j = \infty \\
& y_{50} + \inf_{\mu>0} \frac{\mu}{e} \exp\left(-\frac{y_{50}}{\mu} + \frac{\|\boldsymbol{v}\|_2^2}{2\mu^2} \right) \le r_5 \\
& v_j \ge q_j y_{5j} \ \forall j : q_j < \infty, \qquad y_{5j} \ge 0 \ \forall j : q_j = \infty \\
& v_j \ge -p_j y_{5j} \ \forall j : p_j < \infty, \qquad y_{5j} \le 0 \ \forall j : p_j = \infty \\
& y_{10} + y_{20} + y_{30} + y_{40} + y_{50} = y^0 \\
& \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4 + \boldsymbol{y}_5 = \boldsymbol{y}. \\
& r_i, y_{i0} \in \Re, \boldsymbol{y}_i, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{d}, \boldsymbol{h} \in \Re^N, i = 1, \ldots, 5, \boldsymbol{u}, \boldsymbol{v} \in \Re^N
\end{array}
$$

Distributionally Robust Optimization

- Unified bound on CVaR

$$
\begin{aligned}
\eta_{1-\epsilon}(y_0, \boldsymbol{y}) = \min \quad & r_1 + r_2 + r_3 + r_4 + r_5 \\
\text{s.t.} \quad & y_{10} + \sum_{j:\bar{z}_j<\infty} s_j \bar{z}_j + \sum_{j:\underline{z}_j<\infty} t_j \underline{z}_j \le r_1 \\
& s_j = 0 \ \forall j : \bar{z}_j = \infty, \qquad t_j = 0 \ \forall j : \underline{z}_j = \infty \\
& \boldsymbol{s}, \boldsymbol{t} \ge 0 \\
& \boldsymbol{s} - \boldsymbol{t} = \boldsymbol{y}_1 \\
& y_{20} + (1/\epsilon - 1)\left(\sum_{j:\bar{z}_j<\infty} d_j \bar{z}_j + \sum_{j:\underline{z}_j<\infty} h_j \underline{z}_j \right) \le r_2 \\
& d_j = 0 \ \forall j : \bar{z}_j = \infty, \qquad h_j = 0 \ \forall j : \underline{z}_j = \infty \\
& \boldsymbol{d} - \boldsymbol{h} = -\boldsymbol{y}_2 \\
& \boldsymbol{d}, \boldsymbol{h} \ge 0 \\
& y_{30} + \sqrt{\tfrac{1-\epsilon}{\epsilon}} \|\boldsymbol{\Sigma}^{1/2} \boldsymbol{y}_3\|_2 \le r_3 \\
& y_{40} + \sqrt{-2\ln(\epsilon)} \|\boldsymbol{u}\|_2 \le r_4 \\
& u_j \ge p_j y_{4j} \ \forall j : p_j < \infty, \qquad y_{4j} \le 0 \ \forall j : p_j = \infty \\
& u_j \ge -q_j y_{4j} \ \forall j : q_j < \infty, \qquad y_{4j} \ge 0 \ \forall j : q_j = \infty \\
& y_{50} + \tfrac{1-\epsilon}{\epsilon} \sqrt{-2\ln(1-\epsilon)} \|\boldsymbol{v}\|_2 \le r_5 \\
& v_j \ge q_j y_{5j} \ \forall j : q_j < \infty, \qquad y_{5j} \ge 0 \ \forall j : q_j = \infty \\
& v_j \ge -p_j y_{5j} \ \forall j : p_j < \infty, \qquad y_{5j} \le 0 \ \forall j : p_j = \infty \\
& y_{10} + y_{20} + y_{30} + y_{40} + y_{50} = y^0 \\
& \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4 + \boldsymbol{y}_5 = \boldsymbol{y}. \\
& r_i, y_{i0} \in \Re, \boldsymbol{y}_i, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{d}, \boldsymbol{h} \in \Re^N, i = 1, \ldots, 5, \boldsymbol{u}, \boldsymbol{v} \in \Re^N
\end{aligned}
$$

Distributionally Robust Optimization

- Joint chance constraints
  - All constraints must be satisfied with high probability for all random variables with the same descriptive statistics.
    - Much harder to solve than single chance constraint!!!

$$\inf_{\mathbb{P}\in\mathbb{F}} \mathbb{P}\left(y_j^0 + \boldsymbol{y}_j'\tilde{\boldsymbol{z}} \leq 0, j \in \mathcal{M}\right) \geq 1-\epsilon$$

Distributionally Robust Optimization

- One idea: Union bound

Suppose $\sum_{j \in \mathcal{M}} \epsilon_j \leq \epsilon$

$$\eta_{1-\epsilon_j}(y_j^0, \boldsymbol{y}_j) \leq 0 \qquad j \in \mathcal{M}$$
$$\Downarrow$$
$$\inf_{\mathbb{P} \in \mathbb{F}} \mathbb{P}\left(y_j^0 + \boldsymbol{y}_j' \tilde{\boldsymbol{z}} \leq 0\right) \geq 1 - \epsilon_j \qquad j \in \mathcal{M}$$
$$\Downarrow$$
$$\inf_{\mathbb{P} \in \mathbb{F}} \mathbb{P}\left(y_j^0 + \boldsymbol{y}_j' \tilde{\boldsymbol{z}} \leq 0, j \in \mathcal{M}\right) \geq 1 - \epsilon$$

Distributionally Robust Optimization

- Union bound
  - Bound is good if constraints are independently distributed
  - Bound is weak if constraints are highly correlated.
  - Need to fix ε_j. Sensible choice $\varepsilon_j=\varepsilon/m$
    - How to optimize over ε_j?

Distributionally Robust Optimization

- Can we do better than union bound?
  - Yes!! W. Chen, S., J. Sun and Teo (2007)

$$\gamma_{1-\epsilon}(\boldsymbol{Y},\boldsymbol{a}) = \inf_{\beta, s^0, \boldsymbol{s}} \left(\beta + \frac{1}{\epsilon}\pi\left(s^0 - \beta, \boldsymbol{s}\right) + \sum_{j \in \mathcal{M}} \frac{1}{\epsilon}\pi\left(a_j y_j^0 - s^0, a_j \boldsymbol{y}_j - \boldsymbol{s}\right) \right)$$

where $\boldsymbol{Y} = (y_1^0, \boldsymbol{y}_1, \ldots, y_m^0, \boldsymbol{y}_m)$.

Distributionally Robust Optimization

Suppose there exists $\boldsymbol{a} > 0$, such that

$$\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{a}) \leq 0$$

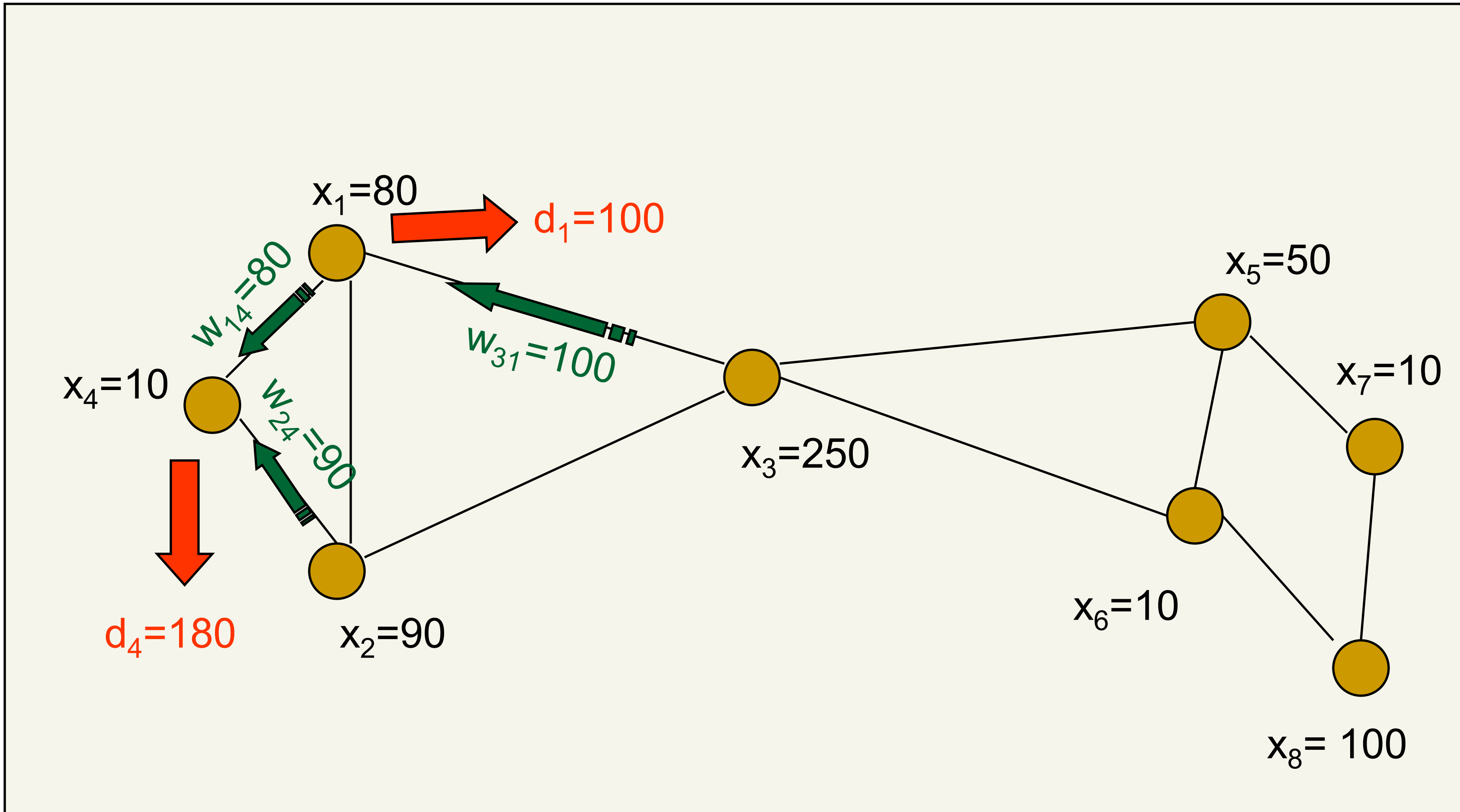
$$\Downarrow$$

$$\inf_{\mathbb{P}\in\mathbb{F}} \mathbb{P}(y_j^0 + \boldsymbol{y}_j'\tilde{\boldsymbol{z}} \leq 0, j \in \mathcal{M}) \geq 1 - \epsilon$$

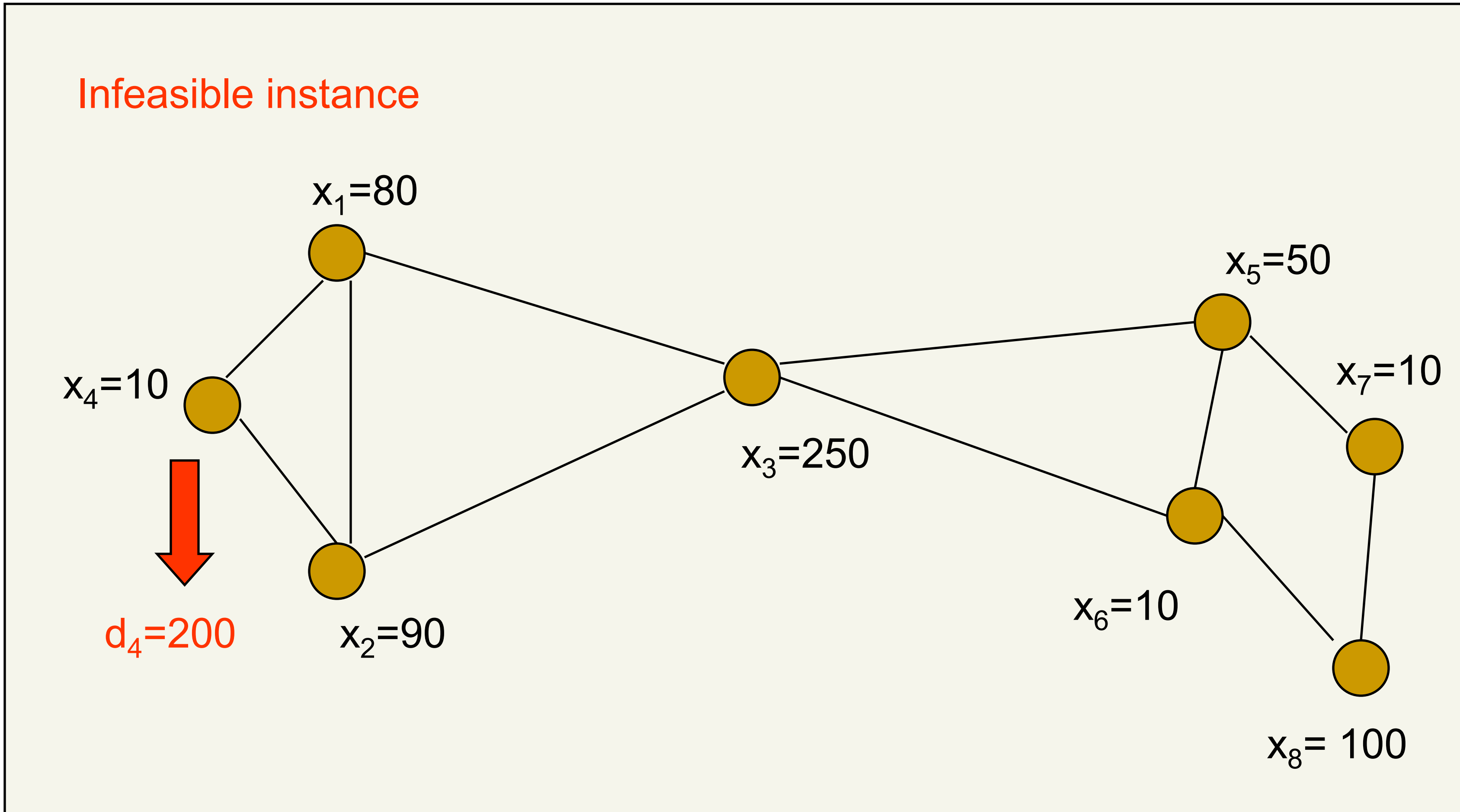
Distributionally Robust Optimization: Resource Allocation Example

- Network of n cities proximally connected
- First stage:
  - Decide amount of resouces to place at each city in anticipation of uncertain demand
- Second stage:
  - Demand is realized
  - Resouces can be transhipped to neighboring nodes at zero cost
- Objective
  - Find the minimum cost allocation of resouces that meets service requirement

Distributionally Robust Optimization: Resource Allocation Example



Distributionally Robust Optimization: Resource Allocation Example



Distributionally Robust Optimization: Resource Allocation Example

- Robust joint chance constrained model

$$
\begin{aligned}
\text{min} \quad & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} \quad & \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{P}\begin{pmatrix} x_i + \sum_{j\in\mathcal{N}^+(i)} w_{ji}(\tilde{\boldsymbol{z}}) - \sum_{j\in\mathcal{N}^-(i)} w_{ij}(\tilde{\boldsymbol{z}}) \geq d_i(\tilde{\boldsymbol{z}}) & i = 1,\ldots,n \\ x_i \geq \sum_{j\in\mathcal{N}^-(i)} w_{ij}(\tilde{\boldsymbol{z}}) & i = 1,\ldots,n \\ \boldsymbol{w}(\tilde{\boldsymbol{z}}) \geq \boldsymbol{0} & \end{pmatrix} \geq 1-\epsilon \\
& \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{w}(\tilde{\boldsymbol{z}})
\end{aligned}
$$

where

$$\mathcal{N}^-(i) \triangleq \{j : (i,j) \in \mathcal{E}\},$$

and

$$\mathcal{N}^+(i) \triangleq \{j : (j,i) \in \mathcal{E}\}.$$

Distributionally Robust Optimization: Resource Allocation Example

- Need to assume linear decision rule on recourse variables on transshipment

$$\boldsymbol{w}(\boldsymbol{z}) = \boldsymbol{w}^0 + \sum_{j=1}^{n} \boldsymbol{w}^j \tilde{z}_j$$

Distributionally Robust Optimization: Resource Allocation Example

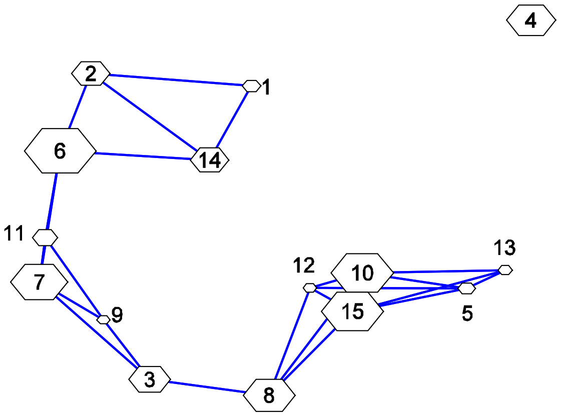
Assume demand at each node are independently distributed with mean 10 and maximum demand of 100. That is,

$$\tilde{d}_j = 10 + \tilde{z}_j$$

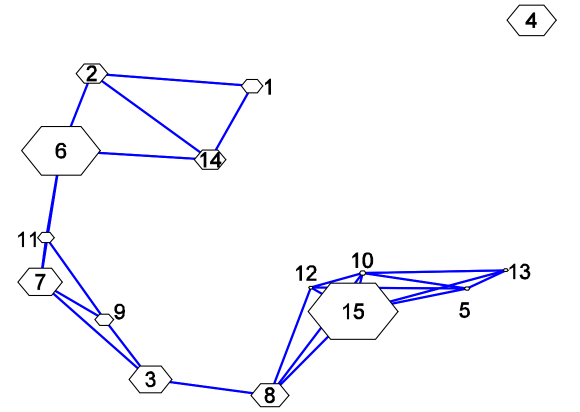
Note that $\sigma_f(\mathbb{F}, \tilde{z}_j) = 42.67$ and $\bar{\sigma}_b(\mathbb{F}, \tilde{z}_j) = 30$.

Distributionally Robust Optimization: Resource Allocation Example

- Computation example: $\varepsilon = 0.01$



Union Bound



Improved Bound

Distributionally Robust Optimization: Resource Allocation Example

- Computation example: $\varepsilon = 0.01$

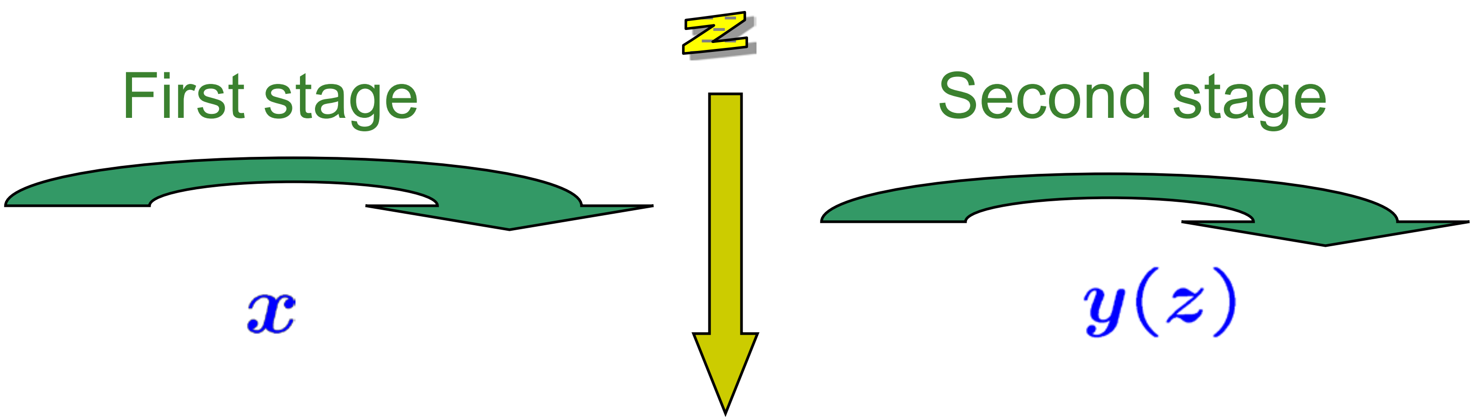
# of Nodes	# of Arcs	Z^W	Z^U	Z^N	$(Z^W - Z^N)/Z^W$	$(Z^U - Z^N)/Z^U$
15	50	1500	1158.1	1043.3	30.45%	9.91%
15	60	1500	1059.7	968.1	35.46%	8.64%
15	70	1500	1027.3	929.5	38.03%	9.52%
15	80	1500	1009.3	890.1	40.66%	11.81%
15	90	1500	989.1	865.7	42.29%	12.48%

Agenda

- Modeling Uncertainty
- DRO
- DRO with Recourse
- Conclusions

DRO with Recourse

- Consider a two stage optimization problem



$$\boldsymbol{c}'\boldsymbol{x} \qquad f(\boldsymbol{x},\boldsymbol{z}) = \min \ \boldsymbol{d}(\boldsymbol{z})'\boldsymbol{y}$$
$$\text{s.t.} \quad \boldsymbol{T}(\boldsymbol{z})\boldsymbol{x} + \boldsymbol{Y}(\boldsymbol{z})\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{z})$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

DRO with Recourse

- Risk Neutral Objective
  - Classical stochastic programming model

$$
\begin{aligned}
Z_{STOC}(\mathbb{P}) = \min \quad & \boldsymbol{c}'\boldsymbol{x} + \mathbb{E}_{\mathbb{P}}(\boldsymbol{d}'\boldsymbol{y}(\tilde{\boldsymbol{z}})) \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \boldsymbol{Y}\boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{h}(\tilde{\boldsymbol{z}}) \\
& \boldsymbol{y}(\tilde{\boldsymbol{z}}) \geq \boldsymbol{0}, \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

  - Assumes repeatability of experiments under identical conditions

DRO with Recourse

- Even when distributions are known, computations can be difficult (Dyer and Stougie, 2005)
  - Two period models are #P-hard
  - >2 periods models are PSPACE-hard

DRO with Recourse

- Ambiguity Averse, Risk Neutral model

$$
\begin{aligned}
Z_{STOC}(\mathbb{F}) = \min \quad & \boldsymbol{c}'\boldsymbol{x} + \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}(\boldsymbol{d}'\boldsymbol{y}(\tilde{\boldsymbol{z}})) \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \boldsymbol{Y}\boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{h}(\tilde{\boldsymbol{z}}) \\
& \boldsymbol{y}(\tilde{\boldsymbol{z}}) \geq \boldsymbol{0}, \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

  - Famous example: Worst case Newsvendor of Scarf.

DRO with Recourse

- Hard problem as well
  - Determine a good upper bound
  - How good is the bound?

$$Z_{STOC}(\mathbb{F}) \leq Z_{ROBUST}$$

DRO with Recourse

- Linear Decision Rule Again!!!
  - Appeared in early Stochastic Optimization but was abandon soon.
    - Garstka and Wets, 74
  - Resurface in adjustable robust counterpart

$$\boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{y}^0 + \sum_{j=1}^{N} \boldsymbol{y}^j \tilde{z}_j$$

DRO with Recourse

- Final Model: Linear Optimization Problem

$$
\begin{aligned}
Z_{LDR} = \quad \min \quad & \boldsymbol{c}'\boldsymbol{x} + \boldsymbol{d}'\boldsymbol{y}^0 \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{T}^k\boldsymbol{x} + \boldsymbol{Y}\boldsymbol{y}^k = \boldsymbol{h}^k \quad \forall k \\
& y_i^0 + \textstyle\sum_{j=1}^N y_i^j z_j \geq 0 \quad \forall \boldsymbol{z} \in \mathcal{W}, \forall i \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

$$Z_{STOC}(\mathbb{F}) \leq Z_{LDR}$$

DRO with Recourse

- Issues with linear decision rule
  - Can lead to infeasible solution even when the problem has complete recourse

$$\begin{array}{ll} \min & \mathsf{E}(y(\tilde{z})) \\ \text{s.t.} & y(z) \geq z \quad \forall z \in \Re \\ & y(z) \geq -z \quad \forall z \in \Re \end{array}$$

Note: $y(z) = |z|$ is feasible.

DRO with Recourse

"The rationale behind restricting to affine decision rules is the belief that in actual applications it is better to pose a modest and achievable goal rather than an ambitious goal which we do not know how to achieve."

- Shapiro and Nemirovski 05

- Can we do better than linear decision rule?

DRO with Recourse

- Exploit problem structure of stocastic optimization model.
  - Focus on recourse matrix Y

$$
\begin{aligned}
Z_{STOC}(\mathbb{F}) = \min \quad & \boldsymbol{c}'\boldsymbol{x} + \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}(\boldsymbol{d}'\boldsymbol{y}(\tilde{\boldsymbol{z}})) \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \boldsymbol{Y}\boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{h}(\tilde{\boldsymbol{z}}) \\
& \boldsymbol{x} \geq \boldsymbol{0} \\
& y_i(\tilde{\boldsymbol{z}}) \geq 0 \ \forall i
\end{aligned}
$$

DRO with Recourse

- Deflected linear decision rule (X. Chen, S., P. Sun and Zhang 2006)

$$\boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{y}^0 + \sum_{j=1}^{N} \boldsymbol{y}^j \tilde{z}_j + \sum_{\{i:\bar{d}_i<\infty\}} \bar{\boldsymbol{y}}^i (-y_i^0 - \boldsymbol{y}_i' \tilde{\boldsymbol{z}})^+$$

$$\boldsymbol{y}_i = (y_i^1, \ldots, y_i^N)$$

DRO with Recourse

where

$$\bar{d}_i = \begin{array}{ll} \min & \boldsymbol{d}'\boldsymbol{y} \\ \text{s.t.} & \boldsymbol{Y}\boldsymbol{y} = \boldsymbol{0} \\ & y_i = 1 \\ & \boldsymbol{y} \geq \boldsymbol{0}, \end{array}$$

An optimum solution: $\bar{\boldsymbol{y}}^i$

DRO with Recourse

- Final Model: SOCP
  - Uses bound on $E(\,.\,)^+$

$$
\begin{aligned}
Z_{DLDR} = \quad \text{min} \quad & \boldsymbol{c}'\boldsymbol{x} + \boldsymbol{d}'\boldsymbol{y}^0 + \sum_{\{i:\bar{d}_i<\infty\}} \bar{d}_i \pi(-y_i^0, -\boldsymbol{y}_i) \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{T}^k\boldsymbol{x} + \boldsymbol{Y}\boldsymbol{y}^k = \boldsymbol{h}^k \quad \forall k \\
& y_i^0 + \sum_{j=1}^{N} y_i^j z_j \geq 0 \quad \forall \boldsymbol{z} \in \mathcal{W}, \forall i : \bar{d}_i = \infty \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

$$
Z_{STOC}(\mathbb{F}) \leq Z_{DLDR} \leq Z_{LDR}
$$

DRO with Recourse

- Primitive uncertainties unfolds in stages

$$
\begin{aligned}
Z_{STOC}(\mathbb{F}) = \min \quad & \boldsymbol{c}'\boldsymbol{x} + \sup_{\mathbb{P}\in\mathbb{F}} \sum_{t=1}^{T} \mathbb{E}_{\mathbb{P}}(\boldsymbol{d}_t'\boldsymbol{y}_t(\boldsymbol{\xi}_t)) \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{T}_t(\boldsymbol{\xi}_t)\boldsymbol{x} + \sum_{\tau=1}^{t} \boldsymbol{Y}_{t\tau}\boldsymbol{y}_\tau(\boldsymbol{\xi}_\tau) = \boldsymbol{b}_t \ \forall t \\
& \boldsymbol{y}_t(\boldsymbol{\xi}_t) \geq \boldsymbol{0} \ \forall t \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

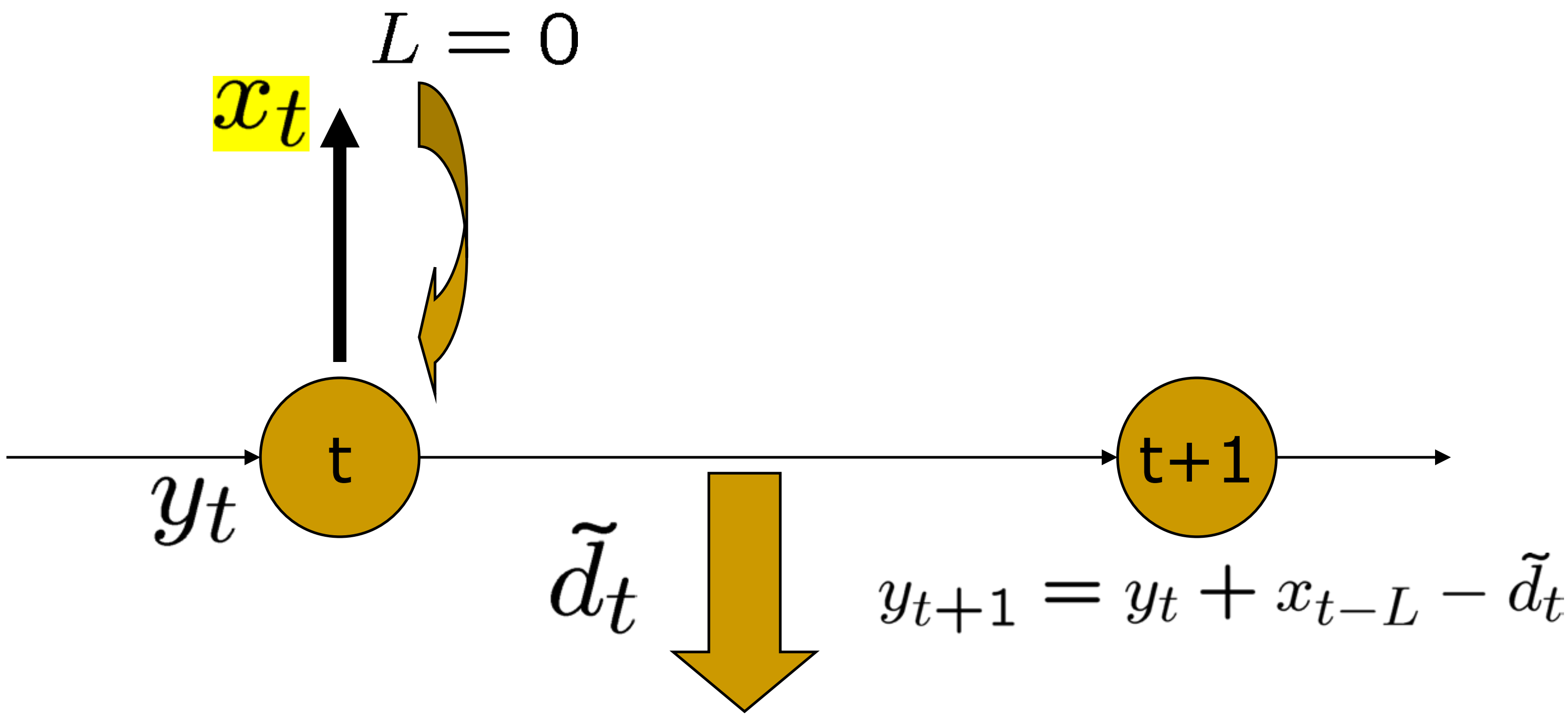
- Scales well with Linear Decsion Rule and Deflected Linear Decision Rules

Robust Inventory Control

- Multiperiod Inventory Control Problem
  - Ordering decision to meet uncertain demand so that the cost is minimized
  - Periodic review, Finite horizon, backlogging, exogenous demand, no fixed ordering costs

Robust Inventory Control

- Sequence of events



Robust Inventory Control

- Inventory dynamics

- $\tilde{d}_t$: stochastic exogenous demand at period t
- $\tilde{\boldsymbol{d}}_t$: a vector of random demands from period 1 to t, that is, $\tilde{\boldsymbol{d}}_t = (\tilde{d}_1, \ldots, \tilde{d}_t)$
- $y_t(\tilde{\boldsymbol{d}}_{t-1})$: net inventory level at the beginning of the tth time period
- $x_t(\tilde{\boldsymbol{d}}_{t-1})$: order placed at the beginning of the tth time period after observing $\tilde{\boldsymbol{d}}_{t-1}$. The first period inventory order is denoted by $x_1(\tilde{\boldsymbol{d}}_0) = x_1^0$

Inventory control constraint:

$$0 \leq x_t(\tilde{\boldsymbol{d}}_{t-1}) \leq S_t$$

Robust Inventory Control

- Costs components

  - c_t: unit purchase cost of inventory for orders placed at the tth time
  - h_t: unit inventory overage (holding) cost charged on excess inventory at the end of the tth time period
  - b_t: unit underage (backlog) cost charged on backlogged inventory at the end of the tth time period

Robust Inventory Control

- Stochastic Optimization Model

$$
\begin{aligned}
Z_{STOC}(\mathbb{P}) = \min \quad & \sum_{t=1}^{T} \left(\mathbb{E}_{\mathbb{P}} \left(c_t x_t(\tilde{\boldsymbol{d}}_{t-1}) \right) + \mathbb{E}_{\mathbb{P}} \left(h_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^+ \right) + \mathbb{E}_{\mathbb{P}} \left(b_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^- \right) \right). \\
\text{s.t.} \quad & y_{t+1}(\tilde{\boldsymbol{d}}_t) = y_t(\tilde{\boldsymbol{d}}_{t-1}) + x_{t-L}(\tilde{\boldsymbol{d}}_{t-L-1}) - \tilde{d}_t \qquad t = 1, \ldots, T \\
& 0 \le x_t(\tilde{\boldsymbol{d}}_{t-1}) \le S_t \qquad t = 1, \ldots, T-L
\end{aligned}
$$

Robust Inventory Control

- Characterize optimum policy using Dynamic Programming
  - Dependent Demand:
    - More realistic representation of demand
    - Curse of dimensionality
  - Independent Demand:
    - State independent base-stock policy is optimal
    - Does not imply that it is easy to find the base-stock level!!

State independent basestock policy:

$$x_t(\tilde{\boldsymbol{d}}_{t-1}) = (q_t - y_t)^+$$

for some base stock-levels, q_t.

Robust Inventory Control

- DRO Inventory Control Model

$$
\begin{aligned}
Z_{STOC}(\mathbb{F}) = \min\ & \sup_{\mathbb{P}\in\mathbb{F}} \sum_{t=1}^{T} \left(\mathbb{E}_{\mathbb{P}}\left(c_t x_t(\tilde{\boldsymbol{d}}_{t-1})\right) + \mathbb{E}_{\mathbb{P}}\left(h_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^{+}\right) + \mathbb{E}_{\mathbb{P}}\left(b_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^{-}\right)\right). \\
\text{s.t. }\ & y_{t+1}(\tilde{\boldsymbol{d}}_t) = y_t(\tilde{\boldsymbol{d}}_{t-1}) + x_{t-L}(\tilde{\boldsymbol{d}}_{t-L-1}) - \tilde{d}_t \qquad t = 1,\ldots,T \\
& 0 \leq x_t(\tilde{\boldsymbol{d}}_{t-1}) \leq S_t \qquad t = 1,\ldots,T-L
\end{aligned}
$$

Robust Inventory Control

- Factor Demand Model

$$d_t(\tilde{z}) \triangleq \tilde{d}_t = d_t^0 + \sum_{k=1}^{N} d_t^k \tilde{z}_k, \qquad t = 1, \ldots, T,$$

where

$$d_t^k = 0 \qquad \forall k \geq N_t + 1,$$

and $1 \leq N_1 \leq N_2 \leq \ldots \leq N_T = N$.
Random factors, $\tilde{z}_k$, $k = 1, \ldots, N$ are realized sequentially.
New factors $\tilde{z}_k$, $k = N_t + 1, \ldots, N_{t+1}$ are made available from period t to $t + 1$.

Robust Inventory Control

- Factor Demand Model
  - Handle demand correlations
  - Can include exogenous factors such as market factors
  - Demand forecast models
    - E.g: ARMA process

$$d_t(\tilde{z}) = \begin{cases} d_t^0 & \text{if } t \leq 0 \\ \sum_{j=1}^{p} \phi_i d_{t-i}(\tilde{z}) + \tilde{z}_t + \sum_{j=1}^{\min\{q,t-1\}} \theta_i \tilde{z}_{t-j} & \text{otherwise} \end{cases}$$

Robust Inventory Control

- Static Replenishment Policy (Bertsimas and Thiele)
  - Inventory position affine in factors

$$x_t^{SRP}(\tilde{\boldsymbol{d}}_{t-1}) = x_t^0 \qquad t = 1,\ldots,T-L$$

$$\begin{array}{lll} Z_{SRP} = \min & \displaystyle\sum_{t=1}^{T}\left(c_t x_t^0 + h_t \pi\left(y_{t+1}^0, \boldsymbol{y}_{t+1}\right) + b_t \pi\left(-y_{t+1}^0, -\boldsymbol{y}_{t+1}\right)\right) & \\ \text{s.t.} & y_{t+1}^0 = y_t^0 + x_{t-L}^0 - d_t^0 & t = 1,\ldots,T \\ & y_{t+1}^k = y_t^k - d_t^k & k = 1,\ldots,N, t = 1,\ldots,T \\ & 0 \le x_t^0 \le S_t & t = 1,\ldots,T-L, \end{array}$$

$$Z_{STOC}(\mathbb{F}) \le Z_{SRP}$$

Robust Inventory Control

- Linear Replenishment Policy

$$x_t^{LRP}(\tilde{\boldsymbol{d}}_{t-1}) = x_t^0 + \boldsymbol{x}_t'\tilde{\boldsymbol{z}} \qquad t = 1, \ldots, T - L$$

in which the vector $\boldsymbol{x}_t = (x_t^1, \ldots, x_t^N)$ satisfies the following non-anticipative constraints,

$$x_t^k = 0 \qquad \forall k \geq N_t.$$

Robust Inventory Control

- Inventory position affine in factors

$$x_t^{LRP}(\tilde{\boldsymbol{d}}_{t-1}) = x_t^0 + \boldsymbol{x}_t'\tilde{\boldsymbol{z}} \qquad t = 1,\ldots,T-L$$

$$\begin{array}{rll} Z_{LRP} = \min & \sum_{t=1}^{T} \left(c_t x_t^0 + h_t \pi\left(y_{t+1}^0, \boldsymbol{y}_{t+1}\right) + b_t \pi\left(-y_{t+1}^0, -\boldsymbol{y}_{t+1}\right)\right) & \\ \text{s.t.} & y_{t+1}^k = y_t^k + x_{t-L}^k - d_t^k & k = 0,\ldots,N, t = 1,\ldots,T \\ & x_t^k = 0 & \forall k \geq N_t, t = 1,\ldots,T-L \\ & 0 \leq x_t^0 + \boldsymbol{x}_t'\boldsymbol{z} \leq S_t & \forall \boldsymbol{z} \in \mathcal{W} \qquad t = 1,\ldots,T-L, \end{array}$$

$$Z_{STOC}(\mathbb{F}) \leq Z_{LRP} \leq Z_{SRP}$$

Robust Inventory Control

- Truncated Linear Replenishment Policy (See and Sim)

$$x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) = \min\left\{\max\left\{x_t^0 + \boldsymbol{x}_t'\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\},$$

in which the vector $\boldsymbol{x}_t = (x_t^1, \ldots, x_t^N)$ satisfies the following non-anticipative constraints,

$$x_t^k = 0 \qquad \forall k \geq N_{t-1} + 1.$$

Observe that

$$0 \leq x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) \leq S_t$$

Net inventory level is not affine in factors!!

Robust Inventory Control

- Require the following bound on expectation:

$$\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left\{\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i=1}^{p}\left(x_i^0 + \boldsymbol{x}_i'\tilde{\boldsymbol{z}}\right)^+\right)^+\right\} \le \eta((y^0,\boldsymbol{y}),(x_1^0,\boldsymbol{x}_1),\ldots,(x_p^0,\boldsymbol{x}_p))$$

where

$$\begin{aligned}&\eta((y^0,\boldsymbol{y}),(x_1^0,\boldsymbol{x}_1),\ldots,(x_p^0,\boldsymbol{x}_p))\\ = \;&\min_{w_i^0,\boldsymbol{w}_i,i=1,\ldots,p}\left\{\pi\left(y^0+\sum_{i=1}^{p}w_i^0,\boldsymbol{y}+\sum_{i=1}^{p}\boldsymbol{w}_i\right)+\sum_{i=1}^{p}\left(\pi(-w_i^0,-\boldsymbol{w}_i)+\pi(x_i^0-w_i^0,\boldsymbol{x}_i-\boldsymbol{w}_i)\right)\right\}.\end{aligned}$$

$\eta((y^0,\boldsymbol{y}),(x_1^0,\boldsymbol{x}_1),\ldots,(x_p^0,\boldsymbol{x}_p))$ is SOC representable function !

Robust Inventory Control

$$x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) = \min\left\{\max\left\{x_t^0 + \boldsymbol{x}_t'\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\},$$

$$\begin{aligned}
Z_{TLRP} = \min \quad & \sum_{t=1}^{T} c_t \pi(x_t^0, \boldsymbol{x}_t) + \sum_{t=1}^{L} (h_t \pi(y_{t+1}^0, \boldsymbol{y}_{t+1}) + b_t \pi(-y_{t+1}^0, -\boldsymbol{y}_{t+1})) + \\
& \quad \sum_{t=L+1}^{T} (h_t \eta\left((y_{t+1}^0, \boldsymbol{y}_{t+1}), (-x_1^0, -\boldsymbol{x}_1), \ldots, (-x_{t-L}^0, -\boldsymbol{x}_{t-L})\right) + \\
& \qquad b_t \eta\left((-y_{t+1}^0, -\boldsymbol{y}_{t+1}), (x_1^0 - S_t, \boldsymbol{x}_1), \ldots, (x_{t-L}^0 - S_t, \boldsymbol{x}_{t-L})\right)) \\
\text{s.t.} \quad & y_{t+1}^k = y_t^k + x_{t-L}^k - d_t^k \qquad k = 0, \ldots, N, t = 1, \ldots, T \\
& x_t^k = 0 \qquad \forall k \geq N_t, t = 1, \ldots, T - L
\end{aligned}$$

$$Z_{STOC}(\mathbb{F}) \leq Z_{TLRP} \leq Z_{LRP} \leq Z_{SRP}$$

Modeling Software

- Sample code implementing TLDR

```
initx =0*ones(L,1);
inity = 0;

Ny=[0 1:T];
Nx = [zeros(1,L) 0:T-L-1];
Nxms = [zeros(1,L) 0:T-L-1];

% Demand information
Z.zlow = Range*ones(N,1);
Z.zupp = Range*ones(N,1);
Z.p = .58*Range*ones(N,1);
Z.q = .58*Range*ones(N,1);
Z.sigma =.58*Range*ones(N,1);
```

Modeling Software

```
startmodel
x = linearrule(T,N,Nx);
xms = linearrule(T,N,Nxms);
y = linearrule(T+1,N,Ny);

for i=1:T
    addconst(xms(i,:) == x(i,:)-S*ldrdata([0 1],N));
end

hbound=0;
sbound=0;
for t=1:T

    if L+1<= t
        hbound = hbound+ h*meannestedposbound(Z,y(t+1,0:t),-x(L+1:t,0:t),t);
        sbound = sbound + b(t)*meannestedposbound(Z,-y(t+1,0:t),xms(L+1:t,0:t),t);
    else
        hbound = hbound+h*meanpositivebound(Z,y(t+1,:),1,N);
        sbound = sbound + b(t)*meanpositivebound(Z,-y(t+1,:),1,N);
    end

end
```

Modeling Software

```
minimize (sbound+hbound + c*sum(meanpositivebound(Z,x(L+1:T,:),T-L,N)))
addconst(x(1:L,0)==initx);
addconst(y(1,0)==inity);
for i=1:T
   addconst(y(i+1,:)==y(i,:)+x(i,:)-ldrdata([0 MeanD(i);(1:N)' zcoef(:,i)],N));
end

m=endmodel;
s = m.solve('CPLEX');
xsol=s.eval(x);
```

Robust Inventory Control
- Computations

Correlated Demand:

$$d_t(\tilde{z}) = \tilde{z}_t + \alpha \tilde{z}_{t-1} + \alpha \tilde{z}_{t-2} + \ldots + \alpha \tilde{z}_1 + \mu,$$

$$d_t(\tilde{z}) = d_{t-1}(\tilde{z}) - (1-\alpha)\tilde{z}_{t-1} + \tilde{z}_t.$$

$\alpha = 0$: iid demand
$\alpha = 1$: random walk

$T = 5,\ \tilde{z}_j \sim U(-20, 20)$

Robust Inventory Control - Computations

- Compare with
  - State independent based-stock policy
    - Ignores dependency of previous demands
    - Policy is optimal if $\alpha = 0$
    - Use sampling approximation to determine reorder point
  - Myopic Policy
    - Ignores future costs

Robust Inventory Control - Computations

- Truncated Linear Decision Rule

Sample solution: $\alpha = 0.4$

$$\begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \\ x_5^0 \end{bmatrix} = \begin{bmatrix} 102.5 \\ 103.0 \\ 101.4 \\ 107.5 \\ 105.6 \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{x}_1' \\ \boldsymbol{x}_2' \\ \boldsymbol{x}_3' \\ \boldsymbol{x}_4' \\ \boldsymbol{x}_5' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1.85 & 0 & 0 & 0 \\ 0.24 & 1.70 & 0 & 0 \\ 0.19 & 0.18 & 1.42 & 0 \\ 0.32 & 0.32 & 0.38 & 1.40 \end{bmatrix}$$

$$x_i^{TLRP}(\boldsymbol{z}) = \min\{(x_i^0 + \boldsymbol{x}_i'\boldsymbol{z})^+, 140\}.$$

Robust Inventory Control - Computations

- Empirical performance (100,000 samples)

α	TLPR	BSP	MP	$\hat{\sigma}$(TLRP)	$\hat{\sigma}$(BSP)	$\hat{\sigma}$(MP)	BSP/TLPR	MP/TLPR
1	2416	3290	2760	5.5	17.1	14.5	1.36	1.14
0.8	2048	2573	2138	2.3	11.2	8.7	1.26	1.04
0.6	1716	2056	1784	1.0	6.1	4.7	1.20	1.04
0.4	1550	1769	1611	0.5	3.4	2.3	1.14	1.04
0.2	1515	1576	1539	0.5	1.2	0.9	1.04	1.02
0	1512	1513	1526	0.4	0.5	0.5	1.00	1.01

Conclusions

- Robust optimization is a computationally attractive approach for addressing data uncertainty in optimization problems
- Many applications
- Many open issues:
  - Quantify level of conservativeness
  - Address non affine disturbances
  - Address general recourse problems
  - Address integral recourse problems