Causal Wiener filtering

Assume that $x$ and $y$ are two jointly stationary second order processes, with spectral density given by

$$\Phi(z) = \begin{bmatrix} \Phi_x & \Phi_{xy} \\ \Phi_{yx} & \Phi_y \end{bmatrix}(z)$$

We would like to find a strictly causal function $F(t)$ such that $\chi(t) = \hat{F}(z)y(t)$ is the best linear estimator of $x(t)$ given $y(t-1), y(t-2), \cdots$.

If $y(t)$ is a white noise process, it is easy to see that the optimal filter is given by a causal function

$$\hat{F}(z) = \begin{bmatrix} \Phi_{xy}(z) \end{bmatrix}_+,$$

where $[\cdot]_+$ denotes the strictly causal part of the function.

**Lemma 4.1.4.** Assume the observation process $y$ is normalized white noise. Then the matrix function $F$ defining the best linear causal estimator $\chi(t)$ of $x(t)$ given the past history of $y$ up to and including time $t-1$, i.e.

$$\chi(t) := E[x(t)|H_{-t}(y)] = \sum_{s=-\infty}^{\infty} F(t-s)y(s)$$

is given by

$$F(t) = \begin{cases} \Lambda_{xy}(t), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

where $\Lambda_{xy}$ is the cross covariance matrix of the processes $x$ and $y$

**Proof:** The orthogonality condition provides

$$E\{(x(t) - \chi(t))y(\tau)'\} = E\{(x(t) - \sum_{s=-\infty}^{\infty} F(t-s)y(s))y(\tau)'\} = 0, \quad \tau \leq t - 1.$$

which can be written as

$$\Lambda_{xy}(t-\tau) = \sum_{s=-\infty}^{\infty} F(t-s)I\delta(t-s) = F(t-\tau), \quad \tau \leq t - 1.$$ 

And in order that $\chi(t) \in H_{-t}(y)$, the function $F$ has to be strictly causal. □

In general, the process $y$ is not white noise, and then the following trick using a whitening filter can be applied. Assuming that we can determine a minimum-phase spectral factor $W$ of $\Phi_y$, the process $e(t) = W^{-1}(z)y(t)$
is the (forward) normalized innovation process of \( y(t) \), i.e. a white noise process and we can consider the cascade form of the Wiener filter illustrated in Figure 1.

The part \( \hat{G} \) can now be determined as \( \hat{F} \) above, the only difference is that we have to exchange \( \Phi_{xy} \) with \( \Phi_{xe} = \Phi_{xy} W^{-*} \).

The estimator is then given by

\[
\chi(t) = \frac{1}{W(z)} \left[ \frac{\Phi_{xy}(z)}{W^*(z)} \right]_+ y(t).
\]

To derive the expression (1), assume that \( \epsilon(t) \) is the innovation process of the joint process \( (x(t), y(t)) \). Then consider the spectral representations

\[
x(t) = \int_{-\pi}^{\pi} e^{it\theta} \hat{x}(\theta) = \int_{-\pi}^{\pi} e^{it\theta} W_x(e^{i\theta}) d\epsilon(\theta).
\]

and

\[
y(t) = \int_{-\pi}^{\pi} e^{it\theta} \hat{y}(\theta) = \int_{-\pi}^{\pi} e^{it\theta} W_y(e^{i\theta}) d\epsilon(\theta),
\]

The cross-covariance \( \Lambda_{xy}(\tau) = \text{E}\{x(t + \tau)y(t)\} \) can be written

\[
\Lambda_{xy}(\tau) = \text{E} \left\{ \int_{-\pi}^{\pi} e^{i(t+\tau)\theta} W_x(e^{i\theta}) d\epsilon(\theta) \int_{-\pi}^{\pi} e^{-i\tau\theta} W_y(e^{-i\theta}) d\epsilon(\theta) \right\}
\]

\[
= \int_{-\pi}^{\pi} e^{i\tau\theta} W_x(e^{i\theta}) W_y(e^{-i\theta}) \frac{d\theta}{2\pi},
\]

and thus \( \Phi_{xy} = W_x W_y^* \). Now, since \( \epsilon(t) = W^{-1}(z)y(t) = W^{-1}(z)W_y(z)e(t) \) it holds that

\[
\Phi_{xe} = W_x(W^{-1}W_y)^* = W_x W_y^* W^{-*} = \Phi_{xy} W^{-*},
\]

which proves (1).
Example 1

Assume that
\[
\Phi_y(z) = \frac{5/4 + 1/2(z + z^{-1})}{5/4 + 1/2(z + z^{-1})}, \quad \Phi_{xy}(z) = \frac{z + \alpha}{z + \beta}
\]
Then, it is easy to see that
\[
\Phi_y(z) = \frac{(z + 1/2)(z^{-1} + 1/2)}{(z - 1/2)(z^{-1} - 1/2)} = W(z)W(z^{-1}),
\]
where
\[
W(z) = \frac{z + 1/2}{z - 1/2}
\]
is stable and minimum-phase.

Then, assuming \(\beta \neq 2\) (simple poles of the denominator below)
\[
\frac{\Phi_{xy}(z)}{W(z^{-1})} = \frac{z + \alpha z^{-1} - 1/2}{z + \beta z^{-1} + 1/2} = \frac{A}{z + \beta} + \frac{B}{z^{-1} + 1/2} + C,
\]
where
\[
A = (\alpha - \beta) \frac{1 + \beta/2}{1 - \beta/2}, \quad B = \frac{1 - \alpha/2}{1 - \beta/2}, \quad C = \frac{1 - \alpha + \beta/2}{1 - \beta/2}.
\]

Now, assuming \(|\beta| < 1\)
\[
\left[ \frac{\Phi_{xy}(z)}{W(z^{-1})} \right]_+ = \frac{A}{z + \beta},
\]
and
\[
\frac{1}{W(z)} \left[ \frac{\Phi_{xy}(z)}{W(z^{-1})} \right]_+ = \frac{z - 1/2}{z + 1/2} \frac{A}{z + \beta}.
\]

Example 2

Consider now a pair \((x, y)\) of jointly stationary stochastic processes, with spectral density given by
\[
\Phi(z) = \begin{bmatrix} \Phi_x & \Phi_{yx} \\ \Phi_{yx} & \Phi_y \end{bmatrix} = \begin{bmatrix} \frac{200+40(z+z^{-1})}{3-(z+z^{-1})} & \frac{1}{z+1/2} \\ \frac{1}{z^{-1}+1/2} & e^{z+z^{-1}} \end{bmatrix}.
\]
The stable minimum phase spectral factor \(W\) of \(\Phi_y\) is given by
\[
W(z) = e^{z^{-1}} = \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} \in H^2.
\]

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It is easy to see that also $W^{-1} \in H^2$.

Then

$$\hat{G}(z) = \left[ \frac{\Phi_{xy}(z)}{W(z^{-1})} \right]_+ = \left[ \frac{1}{z+1/2} \right]_+ = \left[ \frac{e^{-z}}{z+1/2} \right]_+,$$

and the causal part can be determined as follows

$$\hat{G}(z) = e^{1/2} \left[ \frac{e^{-z+1/2}}{z+1/2} \right]_+$$

$$= e^{1/2} \left[ \frac{1 - (z + 1/2) + \frac{1}{2!}(z + 1/2)^2 - \frac{1}{3!}(z + 1/2)^3 + \cdots}{z+1/2} \right]_+$$

$$= e^{1/2} \left( \left[ \frac{1}{z+1/2} \right]_+ - [1]_+ + \left[ \frac{1}{2!}(z + 1/2) \right]_+ - \left[ \frac{1}{3!}(z + 1/2)^2 \right]_+ + \cdots \right)$$

$$= e^{1/2} \frac{1}{z+1/2}.$$

Alternatively, note that $G(t) = 0$ for $t \leq 0$, and for $t > 0$

$$G(t) = \Lambda_{xe}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it\theta} \frac{\Phi_{xy}(e^{i\theta})}{W(e^{-i\theta})} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it\theta} \frac{e^{-e^{i\theta}}}{e^{-i\theta} + 1/2} d\theta.$$

By using the Residue theorem

$$G(t) = z^{t-1} e^{-z} \bigg|_{z=-1/2} = \left( -\frac{1}{2} \right)^{t-1} e^{1/2}, \quad t = 1, 2, \ldots.$$

Then

$$\hat{G}(z) = \sum_{t=1}^{\infty} \left( -\frac{1}{2} \right)^{t-1} e^{1/2} z^{-t} = e^{1/2} z^{-1} (1 + 1/2 z^{-1})^{-1} = \frac{e^{1/2}}{z + 1/2},$$

as before.

The optimal prediction is given by the filter

$$\frac{1}{W(z)} \left[ \frac{\Phi_{xy}(z)}{W(z^{-1})} \right]_+ = \frac{1}{e^{z^{-1}} z + 1/2} = \frac{e^{1/2 - z^{-1}}}{z + 1/2}.$$