CHAPTER 2

Invariant and controlled invariant subspaces

In this chapter we introduce two important concepts: invariant subspace and controlled invariant subspace, which will be used later on to solve many control problems.

2.1. Invariant subspaces

Consider an n-dimensional linear system

$$\dot{x} \;=\; Ax \tag{2.1}$$

where $x \in \mathbb{R}^n$.

DEFINITION 2.1. *A set $\Omega \subseteq \mathbb{R}^n$ is called an* invariant set *of* (2.1) *if for any initial condition $x_0 \in \Omega$, we have $x(x_0, t) = e^{At}x_0 \in \Omega,\ \forall t \geq 0$.*

Some trivial examples of invariant sets are $\mathbb{R}^n$ and $x = \{0\}$.

In this course we only consider a special class of invariant sets: invariant subspaces. Now let us discuss conditions for a subspace $\mathcal{S}$ to be invariant. Since by Taylor expansion we have

$$x(x_0, t) = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0 + \cdots,$$

it is obvious that if $A^i x_0 \in S\ \forall i \geq 0$, then $x(x_0, t) \in S,\ \forall t \geq 0$. Naturally this argument is true only if $\mathcal{S}$ is a linear subspace. It is easy to see as a sufficient condition

$$Az \in \mathcal{S}\ \forall z \in \mathcal{S}. \tag{2.2}$$

In other words, this condition implies that if we define a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$: $w = Az$, then the image of $S \subseteq \mathbb{R}^n$ is contained in $\mathcal{S}$. We denote this by

$$A\mathcal{S} \subseteq \mathcal{S}. \tag{2.3}$$

Now we show this condition is also necessary for S to be invariant.

PROPOSITION 2.1. *A necessary and sufficient condition for a linear subspace $\mathcal{S}$ to be invariant under* (2.1) *is that condition* (2.3) *holds.*

PROOF

We only show the necessity here. Suppose there exists a point $x_0 \in \mathcal{S}$ such that $Ax_0 \notin \mathcal{S}$. Then when t is sufficiently small, we have

$$x(x_0, t) = x_0 + tAx_0 + \mathcal{O}(t^2),$$

which does not belong to $\mathcal{S}$, since $\mathcal{S}$ is closed. ∎

EXAMPLE 2.1. *Consider*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{\cdot} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Show that $S = span\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ *is invariant.*

We first use the definition of invariant set to show this. It is easy to see that the set can be redefined as $S = \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$. Then to show S to be invariant is to show $x_1(x_0, t) + x_2(x_0, t) = 0 \ \forall t \geq 0$ if $x_0 \in S$. This is equivalent to showing $\dot{x}_1 + \dot{x}_2 = 0$ for all $(x_1, x_2)^T \in S$. We have $\dot{x}_1 + \dot{x}_2 = -2x_2 - x_2 + x_1 = -(x_1 + x_2) = 0$ if $(x_1, x_2)^T \in S$.

We can also show this with Proposition 2.1, since $A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

REMARK 2.1. *As an example to show the above result is only true for subspaces, we consider a circle defined by* $R = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$. *It is easy to show (as an exercise) that this set is invariant under the system*

$$\begin{aligned} \dot{x}_1 &= \omega x_2 \\ \dot{x}_2 &= -\omega x_1, \end{aligned}$$

where ω *is any positive number.*

We ask the reader to check if $Az \subset R$ *for any* $z \in R$.

Then we can use condition (2.3) as an alternative definition for invariant subspace.

DEFINITION 2.2. *A linear subspace* $\mathcal{S}$ *is* A-invariant *(invariant under* $\dot{x} = Ax$*) iff* $A\mathcal{S} \subseteq \mathcal{S}$.

2.2. Controlled invariant subspaces

Now we consider a control system

$$\dot{x} = Ax + Bu \tag{2.4}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

DEFINITION 2.3. $\mathcal{S}$ *is called a* controlled invariant subspace *of* (2.4) *if there exists a feedback control* $u = Fx$ *such that* $\mathcal{S}$ *is an invariant subspace of*

$$\dot{x} = (A + BF)x.$$

Similar to invariant subspace, we can also give another equivalent definition.

DEFINITION 2.4. $\mathcal{S}$ *is an* (A, B)*-invariant (controlled invariant) subspace if there exists a matrix* F *such that*

$$(A + BF)\mathcal{S} \subseteq \mathcal{S}. \tag{2.5}$$

Such an F *is called a* **friend** *of* $\mathcal{S}$.

We denote the set of friends by $\mathcal{F}(\mathcal{S})$. The following theorem provides a fundamental characterization of (A, B)-invariant subspaces that removes the explicit involvement of the feedback matrix F.

THEOREM 2.2. *$\mathcal{S}$ is (A, B)-invariant if and only if*

$$A\mathcal{S} \subseteq \mathcal{S} + \text{Im } B. \tag{2.6}$$

PROOF
Necessity: Suppose F is a friend, then

$$(A + BF)\mathcal{S} \subseteq \mathcal{S}.$$

or

$$A\mathcal{S} \subseteq \mathcal{S} - B(F\mathcal{S}).$$

Since $B(F\mathcal{S}) \subseteq \text{Im } B$, thus (2.6) holds.
Sufficiency: The proof is constructive and is given as follows. ∎

We now give an algorithm for finding a friend of $\mathcal{V}$, which also serves as a proof of the sufficiency of Theorem 2.2.

Algorithm for finding F
Let $\{v_1, v_2, \dots, v_r\}$ be a basis for $\mathcal{V}$. Since $\mathcal{V}$ satisfies $A\mathcal{V} \subseteq \mathcal{V} + \text{Im } B$, there is for each $i = 1, \dots, r$ a $w_i \in \mathcal{V}$ and a $u_i \in \mathbb{R}^m$ such that

$$Av_i = w_i + Bu_i.$$

Let F be a $m \times n$-matrix such that $Fv_i = -u_i$ for $i = 1, 2, \dots, r$ (if $r < n$ then F is not unique). Then $Av_i = w_i - BFv_i$, i.e., $(A + BF)v_i = w_i \in \mathcal{V}$ and therefore $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.

EXAMPLE 2.2. *Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Is the subspace $\mathcal{V} = \{x \in \mathbb{R}^2 : x_1 = x_2\}$ (A, B)-invariant? If so, find a friend of $\mathcal{V}$.*

Clearly, $\mathcal{V}$ is spanned by $v = \begin{bmatrix} 1 & 1 \end{bmatrix}'$. Since

$$Av = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathcal{V} + \text{Im } B,$$

$\mathcal{V}$ is (A, B)-invariant, and we can let $u = 2$. To find F we must solve the under-determined system of equations $Fv = -u$, i.e.,

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2.$$

The set $\mathcal{F}(\mathcal{V})$ is the affine space $\{(\lambda - 2, -\lambda); \lambda \in \mathbb{R}\}$. Choose, e.g., $F = \begin{bmatrix} -2 & 0 \end{bmatrix}$.

Is the subspace $\mathcal{V} := \{x \in \mathbb{R}^2 : x_1 = 0\}$ (A, B)-invariant?

2.3. Reachability subspaces

In the rest of this chapter, we study the most elementary class of controlled invariant subspaces: *reachability (controllability) subspace.*

Definition 2.5. *We use the notation $\langle A|S\rangle$ to denote the minimal A-invariant subspace that contains subspace S.*

Naturally if S is already A-invariant, then $\langle A|S\rangle = S$. $\langle A|S\rangle$ can be computed in the following way:

(1) Let $S_0 = S$, check if $AS_0 \subseteq S_0$. If yes, stop. Otherwise,
(2) Let $S_{k+1} = AS_k + S_k$, $k \geq 0$.
(3) Check if $AS_{k+1} \subseteq S_{k+1}$. If yes, stop. Otherwise return to step 2.

Consider again (2.4). Recall that the reachable (controllable) subspace of (2.4) can be defined with our notation as

$$\langle A|\operatorname{Im} B\rangle = span\{B, AB, \cdots, A^{n-1}B\},$$

namely, the minimal A-invariant subspace that contains Im B. However, for many complex control problems, such as the problem of controllability under constraints discussed in the introduction, more refined study of reachability is needed.

Now consider the feedback law

$$u = Fx + Gv. \tag{2.7}$$

The corresponding closed-loop system

$$\dot{x} = (A + BF)x + BGv$$

has the reachable subspace

$$\mathcal{R} = \langle A + BF|\operatorname{Im} BG\rangle. \tag{2.8}$$

Remark 2.2. *By construction, $\mathcal{R}$ is (A, B)-invariant.*

Definition 2.6. *A subspace $\mathcal{R}$ is called a* reachability subspace *of (2.4) if there are F and G such that* (2.8) *holds.*

Example 2.3. *If $G = I$ then*

$$\mathcal{R} = \langle A + BF|\operatorname{Im} B\rangle = \langle A|\operatorname{Im} B\rangle,$$

is the reachable subspace. If $G = 0$ then $\mathcal{R} = 0$. For a SISO*-system it is obvious that these are the only possible reachability subspaces.*

We now proceed with the analysis of reachability subspaces. The first theorem shows that the matrix G can be removed from the characterization of $\mathcal{R}$ at the price of an implicit characterization, which however is of great use.

Theorem 2.3. *A subspace $\mathcal{R}$ is a reachability subspace if and only if there is an F such that*

$$\mathcal{R} = \langle A + BF|\operatorname{Im} B \cap \mathcal{R}\rangle. \tag{2.9}$$

PROOF
Necessity: Suppose $\mathcal{R}$ is a reachability subspace, i.e.,

$$\mathcal{R} = \langle A + BF | \operatorname{Im} BG \rangle \tag{2.10}$$

for some F and G. Then $\operatorname{Im} BG \subseteq \mathcal{R}$ and $\operatorname{Im} BG \subseteq \operatorname{Im} B$, i.e.,

$$\operatorname{Im} BG \subseteq \operatorname{Im} B \cap \mathcal{R}.$$

Hence,

$$\mathcal{R} \subseteq \langle A + BF | \operatorname{Im} B \cap \mathcal{R} \rangle. \tag{2.11}$$

But $\mathcal{R}$ is (A, B)-invariant and therefore

$$(A + BF)^k \mathcal{R} \subseteq \mathcal{R} \text{ for } k \geq 1$$

and

$$\langle A + BF | \operatorname{Im} B \cap \mathcal{R} \rangle \subseteq \mathcal{R}. \tag{2.12}$$

Now (2.9) follows from (2.11) and (2.12).

Sufficiency: Suppose that (2.9) holds. It is enough to show that there is a G such that $\operatorname{Im} B \cap \mathcal{R} = \operatorname{Im} BG$, since this will imply (2.10).

Let $p_1, p_2, \ldots, p_q$ be a basis for $\operatorname{Im} B \cap \mathcal{R}$. Then there is a linearly independent set $\{u_1, u_2, \ldots, u_q\}$ such that

$$p_i = Bu_i \qquad i = 1, 2, \ldots, q,$$

since if the u_i's were linearly dependent then the p_i's would be linearly dependent as well. If we let the input space be R^m it holds that $q \leq \dim(\operatorname{Im} B) \leq m$. Choose $u_{q+1}, \ldots, u_m$ such that $\{u_1, \ldots, u_m\}$ is a basis for $\mathbb{R}^m$. We want

$$BGu_i = \begin{cases} p_i & i = 1, 2, \ldots, q, \\ 0 & i = q + 1, \ldots, m \end{cases}$$

which yields $\operatorname{Im} BG = \operatorname{Im} B \cap \mathcal{R}$, i.e.,

$$BG[u_1, \ldots, u_m] = [p_1, \ldots, p_q, 0, \ldots, 0] = B[u_1, \ldots, u_q, 0, \ldots, 0].$$

This is achieved by

$$G := [u_1, \ldots, u_q, 0, \ldots, 0][u_1, u_2, \ldots, u_m]^{-1}.$$

∎

We know that the reachability subspace $\mathcal{R}$ is (A, B)-invariant and it is obvious that F in

$$\mathcal{R} = \langle A + BF | \operatorname{Im} B \cap \mathcal{R} \rangle \tag{2.13}$$

is a friend of $\mathcal{R}$, i.e., $F \in \mathcal{F}(\mathcal{R})$.

EXAMPLE 2.4. *Consider*

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= x_1 + u_2. \end{aligned}$$

We will show that $V = span\{e_1, e_2\}$ is a reachability subspace.

It is easy to compute that $F = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ *is a friend of* V, *and* $ImB \cap V = span\{e_2\}$. *Then, it is easy to calculate that* $(A+BF)e_2 = e_1$ *and* $(A+BF)e_1 = e_1$. *Thus* $< A + BF|ImB \cap V >= V$. *We note that* $V_1 = span\{e_1\}$ *is an (A,B)-invariant subspace but not a reachability subspace since* $ImB \cap V_1 = 0$.

The next theorem shows that the representation (2.13) is independent of the actual choice of $F \in \mathcal{F}(\mathcal{R})$.

THEOREM 2.4. *Let* $\mathcal{R}$ *be a reachability subspace and let* $\hat{F} \in \mathcal{F}(\mathcal{R})$, *i.e. an arbitrary friend of* $\mathcal{R}$. *Then*

$$\mathcal{R} = \langle A + B\hat{F} | \text{ Im } B \cap \mathcal{R} \rangle.$$

PROOF
From Theorem 2.3 follows the existence of an F such that

$$\mathcal{R} = \langle A + BF | \text{ Im } B \cap \mathcal{R} \rangle.$$

Now let $\hat{F} \in \mathcal{F}(\mathcal{R})$ be an arbitrary friend and form

$$\hat{\mathcal{R}} = \langle A + B\hat{F} | \text{ Im } B \cap \mathcal{R} \rangle.$$

Since $(A + B\hat{F})\mathcal{R} \subseteq \mathcal{R}$, it holds that $\hat{\mathcal{R}} \subseteq \mathcal{R}$.

We shall show that $R \subseteq \hat{\mathcal{R}}$ by induction. Clearly Im $B \cap \mathcal{R} \subseteq \hat{\mathcal{R}}$. Assume that

$$(A + BF)^k(\text{Im } B \cap \mathcal{R}) \subseteq \hat{\mathcal{R}}.$$

Then

$$\begin{aligned} (A+BF)^{k+1}(\text{Im } B \cap \mathcal{R}) &\subseteq (A+BF)\hat{\mathcal{R}} \\ &\subseteq (A+B\hat{F})\hat{\mathcal{R}} + B(F-\hat{F})\hat{\mathcal{R}} \\ &\subseteq \hat{\mathcal{R}} \text{ if } B(F-\hat{F})\hat{\mathcal{R}} \subseteq \hat{\mathcal{R}} \end{aligned}$$

If we can show the last inclusion, then it follows by induction that $R \subseteq \hat{\mathcal{R}}$.

We need to introduce a lemma here in order to carry on the proof.

LEMMA 2.5. *Let* $F_1 \in \mathcal{F}(\mathcal{V})$. *Then* $F_2 \in \mathcal{F}(\mathcal{V})$ *if and only if* $B(F_1 - F_2)\mathcal{V} \subseteq \mathcal{V}$.

PROOF (Proof of the lemma)
(only if) Suppose $F_1, F_2 \in \mathcal{F}(\mathcal{V})$. Then for all $v \in \mathcal{V}$ it holds that $(A + BF_1)v \in \mathcal{V}$ and $(A + BF_2)v \in \mathcal{V}$, which implies that $B(F_1 - F_2)v \in \mathcal{V}$.
(if) Let $v \in \mathcal{V}$. Then $(A + BF_1)v + B(F_2 - F_1)v = (A + BF_2)v \in \mathcal{V}$, since the terms on the left hand side are in $\mathcal{V}$. ∎

Now we return to the proof of the theorem. Since $\hat{\mathcal{R}} \subseteq \mathcal{R}$ it holds that

$$B(F - \hat{F})\hat{\mathcal{R}} \subseteq B(F - \hat{F})\mathcal{R} \subseteq \{\text{Lemma 2.5}\} \subseteq \mathcal{R}.$$

But $B(F-\hat{F})\hat{\mathcal{R}} \subseteq \operatorname{Im} B$ and therefore

$$B(F-\hat{F})\hat{\mathcal{R}} \subseteq \operatorname{Im} B \cap \mathcal{R} \subseteq \hat{\mathcal{R}}.$$

∎

Combining Theorem 2.3 and Theorem 2.4 we obtain the following result, which can be used to test whether a given subspace is a reachability subspace.

COROLLARY 2.6. *Suppose $\mathcal{V}$ is (A,B)-invariant and let $F \in \mathcal{F}(\mathcal{V})$ be an arbitrary friend of $\mathcal{V}$. The necessary and sufficient condition for $\mathcal{V}$ to be a reachability subspace is that*

$$\langle A+BF | \operatorname{Im} B \cap \mathcal{V}\rangle = \mathcal{V}.$$

.

2.4. Maximal reachability subspaces

Consider the class $S(\mathcal{Z})$ of (A,B)-invariant subspaces contained in $\mathcal{Z}$, and in particular reachability subspaces $\mathcal{R}$ such that $\mathcal{R} \in S(\mathcal{Z})$. All these satisfy

$$\mathcal{R} \subseteq \mathcal{S}^*(\mathcal{Z}), \tag{2.14}$$

where $\mathcal{S}^*(\mathcal{Z})$ is the maximal (A,B)-invariant subspace in $\mathcal{Z}$. The existence of $\mathcal{S}^*$ is shown as follows.

LEMMA 2.7. *Let $\mathcal{Z}$ be a subspace of $\mathbb{R}^n$. Then, the class $S(\mathcal{Z})$ of all (A,B)-invariant subspaces $\mathcal{S} \subseteq \mathcal{Z}$ has a maximal element $\mathcal{S}^*(\mathcal{Z})$ in the sense that*

$$\mathcal{S} \subseteq \mathcal{S}^*(\mathcal{Z}) \text{ for all } \mathcal{S} \in S(\mathcal{Z}).$$

PROOF
Note first that the set $S(\mathcal{Z})$ is closed under addition, i.e., if $\mathcal{S}_1, \mathcal{S}_2 \in S(\mathcal{Z})$, then $\mathcal{S}_1 + \mathcal{S}_2 \subseteq \mathcal{Z}$ and

$$A(\mathcal{S}_1+\mathcal{S}_2) = A\mathcal{S}_1 + A\mathcal{S}_2 \subseteq \mathcal{S}_1+\mathcal{S}_2+\operatorname{Im} B.$$

Hence, $\mathcal{S}_1+\mathcal{S}_2 \in S(\mathcal{Z})$.

Since $\mathcal{Z}$ is of finite dimension, there is an element $\mathcal{S}^* \in S(\mathcal{Z})$ of largest dimension. If $\mathcal{S} \in S(\mathcal{Z})$, then $\mathcal{S}+\mathcal{S}^* \in S(\mathcal{Z})$ and $\mathcal{S}^* \subseteq \mathcal{S}+\mathcal{S}^*$. However, $\mathcal{S}^*$ has maximal dimension and therefore, $\dim(\mathcal{S}+\mathcal{S}^*) = \dim \mathcal{S}^*$, and then, $\mathcal{S}^* = \mathcal{S}+\mathcal{S}^*$, that is, $\mathcal{S} \subseteq \mathcal{S}^*$. Thus, $\mathcal{S}^*$ is maximal in terms of subspace inclusion. ∎

Is there also a *maximal* $\mathcal{R}$ that satisfies (2.14)? *Maximal* in the sense that it contains all other such reachability subspaces.

THEOREM 2.8. *Let $\mathcal{S}^*$ be the maximal (A,B)-invariant subspace in $\mathcal{Z}$, and let $F \in \mathcal{F}(\mathcal{S}^*)$. Then the maximal reachability subspace in $\mathcal{Z}$ is*

$$\mathcal{R}^* := \langle A+BF | \operatorname{Im} B \cap \mathcal{S}^*\rangle. \tag{2.15}$$

Moreover, $F \in \mathcal{F}(\mathcal{R}^*)$*, i.e.,*

$$\mathcal{F}(\mathcal{R}^*) \supseteq \mathcal{F}(\mathcal{S}^*).$$

We develop the proof with the help of the following two lemmas. The first of them is a refinement of Theorem 2.4, where we learn that a reachability subspace can in fact be characterized by any friend of the smaller class of friends of an (A, B)-invariant subspace which generates the reachability subspace in a specific way.

LEMMA 2.9. *Let* $\mathcal{S}$ *be* (A, B)*-invariant and let*

$$\mathcal{R} := \langle A + BF | \hat{\mathcal{B}} \rangle,$$

where $F \in \mathcal{F}(\mathcal{S})$ *and* $\hat{\mathcal{B}} = \operatorname{Im} B \cap \mathcal{S}$*. If* $\hat{F}$ *is any matrix such that* $B(\hat{F} - F)\mathcal{S} \subset \mathcal{S}$ *then we also have that* $\mathcal{R} = \langle A + B\hat{F} | \hat{\mathcal{B}} \rangle$*.*

REMARK 2.3. *Recalling Lemma 2.5 we see that the condition for* $\hat{F}$ *in the above lemma amounts to* $\hat{F} \in \mathcal{F}(\mathcal{S})$*.*

PROOF
Let

$$\hat{\mathcal{R}} := \langle A + B\hat{F} | \hat{\mathcal{B}} \rangle$$

and

$$\mathcal{S}_i := \hat{\mathcal{B}} + (A + BF)\hat{\mathcal{B}} + \ldots + (A + BF)^{i-1}\hat{\mathcal{B}}.$$

Then $\mathcal{S}_1 \subseteq \hat{\mathcal{R}}$.

Proceeding by induction, assume that $\mathcal{S}_i \subseteq \hat{\mathcal{R}}$. Then

$$\mathcal{S}_{i+1} = \hat{\mathcal{B}} + (A + BF)\mathcal{S}_i \subseteq \hat{\mathcal{B}} + (A + B\hat{F})\mathcal{S}_i + B(F - \hat{F})\mathcal{S}_i,$$

which is included in $\hat{\mathcal{R}}$ if

$$B(F - \hat{F})\hat{\mathcal{R}} \subseteq \hat{\mathcal{R}}. \tag{2.16}$$

If so, $\mathcal{R} = \mathcal{S}_n \subseteq \hat{\mathcal{R}}$ by induction.

We now show (2.16). Since $\hat{F} \in \mathcal{F}(\mathcal{S})$ and $\hat{\mathcal{B}} \subseteq \mathcal{S}$ it follows that $\hat{\mathcal{R}} \subseteq \mathcal{S}$. Therefore,

$$B(F - \hat{F})\hat{\mathcal{R}} \subseteq B(F - \hat{F})\mathcal{S} \subseteq \hat{\mathcal{B}} \subseteq \hat{\mathcal{R}}$$

and (2.16) follows. We have thus shown that $\mathcal{R} \subseteq \hat{\mathcal{R}}$.

If we interchange F and $\hat{F}$ in the calculations above, we get $\hat{\mathcal{R}} \subseteq \mathcal{R}$. ∎

LEMMA 2.10. *Let* $\mathcal{R}$ *and* $\mathcal{S}$ *be* (A, B)*-invariant, and suppose that* $\mathcal{R} \subseteq \mathcal{S}$*. Then, if* $\hat{F} \in \mathcal{F}(\mathcal{R})$*, there is an* $F \in \mathcal{F}(\mathcal{R}) \cap \mathcal{F}(\mathcal{S})$ *such that* $F|_{\mathcal{R}} = \hat{F}|_{\mathcal{R}}$*.*

PROOF
Let $\mathcal{W}$ be a subspace such that

$$\mathcal{R} \oplus \mathcal{W} = \mathcal{S}.$$

Let $\{w_1, \ldots, w_q\}$ be a basis for $\mathcal{W}$. Since $\mathcal{S}$ is (A, B)-invariant and $\mathcal{W} \subseteq \mathcal{S}$, we have

$$Aw_i = v_i + Bu_i$$

for some $v_i \in \mathcal{S}$ and $u_i \in \mathbb{R}^m$. Now let $F : \mathbb{R}^n \to \mathbb{R}^m$ be such that $Fx = \hat{F}x$ for $x \in \mathcal{R}$ and $Fw_i = -u_i$. Then $F|_{\mathcal{R}} = \hat{F}|_{\mathcal{R}}$ and $(A + BF)\mathcal{R} \subseteq \mathcal{R} \subseteq \mathcal{S}$. Moreover, $(A + BF)w_i = v_i$, i.e., $(A + BF)\mathcal{W} \subseteq \mathcal{S}$. Hence, $(A + BF)\mathcal{S} \subseteq \mathcal{S}$. ∎

We now prove Theorem 2.8.

PROOF
We need to show that $\mathcal{R}^*$ as defined by $\langle A + BF|\,\mathrm{Im}\ B \cap \mathcal{S}^*\rangle$ where F is any friend of $\mathcal{S}^*$, is a reachability subspace in $\mathcal{Z}$, and moreover that it is maximal.

Since $\mathrm{Im}\ B \cap \mathcal{S}^* \subseteq \mathcal{S}^*$ we have

$$\mathcal{R}^* \subseteq \langle A + BF|\,\mathcal{S}^*\rangle = \mathcal{S}^* \subseteq \mathcal{Z}$$

and we can always choose G such that $\mathrm{Im}\ BG = \mathrm{Im}\ B \cap \mathcal{S}^*$. So $\mathcal{R}^*$ is a reachability subspace in Z.

Next we show that $\mathcal{R} \subset \mathcal{R}^*$ for all reachability subspaces contained in $\mathcal{Z}$. If $\mathcal{R}$ is an arbitrary reachability subspace in $\mathcal{Z}$, it can be expressed as

$$\mathcal{R} = \langle A + BF_0|\,\mathrm{Im}\ B \cap \mathcal{R}\rangle$$

for some $F_0 \in \mathcal{F}(\mathcal{R})$. Clearly, $\mathcal{R} \subseteq \mathcal{S}^*$. Moreover, by Lemma 2.10 there is an $F_1 \in \mathcal{F}(S^*)$ such that

$$F_1|_{\mathcal{R}} = F_0|_{\mathcal{R}}. \tag{2.17}$$

Now if $x \in \mathcal{S}^*$ then

$$B(F - F_1)x = (A + BF)x - (A + BF_1)x \in \mathcal{S}^*,$$

since $F, F_1 \in \mathcal{F}(\mathcal{S}^*)$. Hence,

$$B(F - F_1)\mathcal{S}^* \subseteq \mathrm{Im}\ B \cap \mathcal{S}^*. \tag{2.18}$$

Consequently,

$$\begin{aligned}
\mathcal{R} &= \langle A + BF_0|\,\mathrm{Im}\ B \cap \mathcal{R}\rangle && (2.19)\\
&= \langle A + BF_1|\,\mathrm{Im}\ B \cap \mathcal{R}\rangle && (2.20)\\
&\subseteq \langle A + BF_1|\,\mathrm{Im}\ B \cap \mathcal{S}^*\rangle && (2.21)\\
&= \langle A + BF|\,\mathrm{Im}\ B \cap \mathcal{S}^*\rangle = \mathcal{R}^*, && (2.22)
\end{aligned}$$

where (2.20) follows from (2.17), (2.21) follows from $\mathcal{R} \subseteq \mathcal{S}^*$, and (2.22) follows from Lemma 2.9 and (2.18). But $\mathcal{R}$ is arbitrary, and therefore $\mathcal{R}^*$ is the unique maximal reachability subspace in $\mathcal{Z}$. ∎

We conclude this section with an example.

EXAMPLE 2.5. *Compute $\mathcal{R}^*$ contained in $\mathcal{Z} = ker\ C$ for*

$$A = \begin{bmatrix} 1 & 2 & 0\\ 0 & 3 & 0\\ 0 & 2 & 3\end{bmatrix}, B = \begin{bmatrix} 0 & 1\\ 2 & 0\\ 3 & 1\end{bmatrix} \ \textit{and}\ C = \begin{bmatrix} 1 & 0 & 0\end{bmatrix}.$$

For this purpose, we need to compute $\mathcal{S}^*$ *first. Set* $\mathcal{S}_0 = \ker C$*, i.e.,*

$$S_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and we check if

$$AS_0 \subset S_0 + ImB.$$

Since

$$[S_0, B] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

has full dimension, clearly,

$$AS_0 \subset S_0 + ImB.$$

Hence,

$$\mathcal{S}^* = \ker C = \mathrm{Im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = [v_1, v_2].$$

(What should we do if $S_0 \neq \mathcal{S}^*$*?) The next step is to determine a friend for* $\mathcal{S}^*$*. Since* $(A + BF)\mathcal{S}^* \subseteq \mathcal{S}^*$ *implies*

$$A\mathcal{S}^* \subseteq \mathcal{S}^* + B(-F)\mathcal{S}^*,$$

Therefore, we form

$$A[v_1, v_2] = \begin{bmatrix} 2 & 0 \\ 3 & 0 \\ 2 & 3 \end{bmatrix} \tag{2.23}$$

$$= \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}. \tag{2.24}$$

The first term in (2.24) *is in* $\mathcal{S}^*$ *and the second term has the form* $B[u_1, u_2]$*. Hence, to find* F *we must solve the system*

$$-F[v_1, v_2] = [u_1, u_2],$$

i.e.,

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \text{ with solution } \begin{bmatrix} f_{11} & 0 & 0 \\ f_{21} & -2 & 0 \end{bmatrix}.$$

If we choose $f_{11} = f_{21} = 0$ *then* $A + BF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$*. Finally, straightforward computations yields*

$$\mathrm{Im}\ B \cap \mathcal{S}^* = \mathrm{Im} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \mathcal{R}^*.$$

2.5. Reachability under state constraints

Consider the system

$$\dot{x} = Ax + Bu \tag{2.25}$$

In this section we shall answer the following question. Let $\mathcal{Z}$ be an arbitrary subspace of $\mathbb{R}^n$. Which states can be reached from the origin if we require that the trajectory lies in $\mathcal{Z}$?

Before giving the main result we state some lemmas.

LEMMA 2.11. *Let $x(t,u)$ be the solution of a controlled differential equation and let $\mathcal{M}$ be a subspace of $\mathbb{R}^n$. If $x(t,u) \in \mathcal{M}$ for all t then $\dot{x}(t,u) \in \mathcal{M}$ for all t.*

The proof is left as an exercise for the reader.

LEMMA 2.12. *Consider the system (2.25) and let $\mathcal{Z}$ be a subspace of $\mathbb{R}^n$. If $x(t) \in \mathcal{Z}$ for $t \geq 0$ then $x(t) \in \mathcal{S}^*(\mathcal{Z})$ for $t \geq 0$.*

The proof is left as an exercise for the reader.

LEMMA 2.13. *Consider the system (2.25) and let $\mathcal{Z}$ be a subspace of $\mathbb{R}^n$. If $x(0) = 0$ and $x(t) \in \mathcal{Z}$ for $t \geq 0$ then $x(t) \in \mathcal{R}^*(\mathcal{Z})$ for $t \geq 0$.*

PROOF
By Lemma 2.12 we know that $x(t) \in \mathcal{S}^*(\mathcal{Z})$ for $t \geq 0$. Now, let F be a friend of $\mathcal{S}^*(\mathcal{Z})$ and write the input as

$$u = Fx + v.$$

Then

$$Bv(t) = \dot{x}(t) - (A + BF)x(t) \in \mathcal{S}^*(\mathcal{Z}) \text{ for } t \geq 0,$$

by Lemma 2.11 and Lemma 2.12. Hence,

$$Bv(t) \in \text{Im } B \cap \mathcal{S}^*(\mathcal{Z}),$$

which implies that

$$x(t) = \int_0^t e^{(A+BF)(t-s)} Bv(s)\, ds \in \langle A + BF | \text{ Im } B \cap \mathcal{S}^*(\mathcal{Z}) \rangle = \mathcal{R}^*(\mathcal{Z}).$$

∎

We can now solve the problem of reachability under constraints.

THEOREM 2.14. *Let $\mathcal{S}$ be the set of states that can be reached from the origin with trajectories in $\mathcal{Z}$, i.e.,*

$$S := \{x \in \mathcal{Z} \mid \exists\, t_1 : x(t_1) = x \text{ and } x(t) \in \mathcal{Z}\, \forall t \in [0, t_1]\}. \tag{2.26}$$

Then,

$$\mathcal{R}^*(\mathcal{Z}) = S.$$

Proof

We first show that $\mathcal{S} \subseteq \mathcal{R}^*(\mathcal{Z})$. Let $x \in \mathcal{S}$. Then there is an input and a time t_1 such that $x(t) \in \mathcal{Z}$ and $x(t_1) = x$. By Lemma 2.13 it follows that $x(t) \in \mathcal{R}^*(\mathcal{Z})$ for $t \leq t_1$, and in particular, $x(t_1) \in \mathcal{R}^*(\mathcal{Z})$.

Next we show that $\mathcal{R}^*(\mathcal{Z}) \subseteq \mathcal{S}$. Let F and G be such that $\mathcal{R}^*(\mathcal{Z})$ is the reachable subspace of the closed-loop system

$$\dot{x} = (A + BF)x + BGv.$$

Clearly, all points in $\mathcal{R}^*(\mathcal{Z})$ can be reached by trajectories in $\mathcal{R}^*(\mathcal{Z})$. Hence, $\mathcal{R}^*(\mathcal{Z}) \subseteq \mathcal{S}$. ∎