CHAPTER 7

Output regulation and internal model principle

Consider a MIMO system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{7.1}$$

An important control problem is to design a controller such that the output of the closed-loop system asymptotically tracks a reference signal. In the literature this problem is called the *servo problem.* Another important problem is the *regulation of the output* to zero, regardless of external disturbances and the initial state. In this chapter we treat these two classical problems in an integrated fashion.

Let us start with an example.

EXAMPLE 7.1. *Consider*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a_1 x_1 + a_2 x_2 + bu + dw_1 \\ y &= x_1, \end{aligned}$$

where the disturbance w_1 *is unknown but constant. We want the output* y *to track a sinusoidal signal* $y_d = \beta \sin(\omega t)$*, while rejecting the disturbance.*

For this system $\mathcal{V}^* = 0$*, thus DDP does not have a solution. On the other hand, we know something about the disturbance in this case (a constant). In particular, we can consider* w_1 *as being generated by the following system:*

$$\dot{w}_1 = 0.$$

Similarly, y_d *can be generated by*

$$\begin{aligned} \dot{w}_2 &= \omega w_3 \\ \dot{w}_3 &= -\omega w_2 \\ y_d &= w_2, \end{aligned}$$

where one chooses the right initial state to make the amplitude equal to β*. In general, all reference or disturbance signals that are Bohl functions (including step, ramp or sinusoid) can be generated by such a dynamical model. Such a model is called an* exo-system*. Note that in the exo-system for the sinusoid, the frequency is fixed, but the amplitude and phase are not.*

Now we can incorporate the exo-systems into the plant model and obtain

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= a_1x_1 + a_2x_2 + bu + dw_1\\ \dot{w}_1 &= 0\\ \dot{w}_2 &= \omega w_3\\ \dot{w}_3 &= -\omega w_2\\ e &= x_1 - w_2,\end{aligned}$$

where e is the tracking error. Then the problem becomes to design a controller for the aggregated system such that $e \to 0$.

Now we are ready to define the output regulation problem in general. Consider a MIMO plant

$$\begin{aligned}\dot{x} &= Ax + Bu + Pw\\ \dot{w} &= Sw\\ e &= Cx - Qw,\end{aligned} \tag{7.2}$$

where w models both the reference signal to track (think of w_2 in the example) and the disturbance to reject (think of w_1 in the example), and e is the tracking error. In this chapter we consider two types of output regulation problems.

1. Full information output regulation problem. Given a system (7.2), find a controller $u = Kx + Ew$, such that

a. $x = 0$ of

$$\dot{x} = (A + BK)x$$

is asymptotically stable;

b. For all initial states, $\lim_{t\to\infty} e(t) = 0$.

2. Error feedback output regulation problem. Given a system (7.2), find a controller

$$\begin{aligned}\dot{z} &= Fz + Ge\\ u &= Hz\end{aligned} \tag{7.3}$$

such that

a. $(x, z) = (0, 0)$ of

$$\begin{aligned}\dot{x} &= Ax + BHz\\ \dot{z} &= GCx + Fz\end{aligned} \tag{7.4}$$

is asymptotically stable;

b. For all initial states, $\lim_{t\to\infty} e(t) = 0$.

REMARK 7.1. *In both cases, condition a) implies that the closed-loop system is asymptotically stable when w is set to zero.*

We note that the solution to the full information problem makes sense in practice only when the plant is free from disturbances. However, it lays the foundation to that of the error feedback problem. We also note that if w tends to zero or if S is stable, then the problems are reduced to the stabilization problems. Thus we typically assume that w is *antistable*. This can be defined as

DEFINITION 7.1. *A system*

$$\dot{w} = Sw$$

is called antistable *if S does not have any eigenvalue with strictly negative real part.*

7.1. Full information output regulation

THEOREM 7.1. *Suppose (A, B) is stabilizable and S is antistable in* (7.2)*. Then the full information output regulation is solvable if and only if the linear matrix equations (Sylvester)*

$$\begin{aligned} \Pi S &= A\Pi + P + B\Gamma \\ 0 &= C\Pi - Q \end{aligned} \tag{7.5}$$

are solved by some Π and Γ.

Note that in the first equation, the model of the exo-system (the matrix S) is used. This suggests that without the incorporation of such a model, one can not in general solve the output regulation problem. This fact is generally known as the *internal model principle.*

PROOF
Sufficiency: Suppose K is such that $A + BK$ is stable. We show

$$u = K(x - \Pi w) + \Gamma w$$

where Π is from equation (7.5), solves the full information problem. Plug in the controller to (7.2)

$$\begin{aligned} \dot{x} &= (A + BK)x + (P - BK\Pi + B\Gamma)w \\ \dot{w} &= Sw \\ e &= Cx - Qw. \end{aligned} \tag{7.6}$$

Apparently requirement a) is fulfilled. Then we know from the previous chapter (Proposition 6.2) that x tends to an invariant subspace $x = \bar{\Pi}w$ where $\bar{\Pi}$ is defined by

$$\bar{\Pi}S = (A + BK)\bar{\Pi} + P - BK\Pi + B\Gamma.$$

It is obvious that $\bar{\Pi} = \Pi$ is a solution. Since S and $A + BK$ do not have any common eigenvalue, the solution is also unique (see, for example, [**4**]). Thus, in steady state we have

$$e = C\bar{\Pi}w - Qw = C\Pi w - Qw = 0.$$

Thus e tends to zero.

Necessity: By requirement a), any controller $u = Kx + Ew$ that solves the full information problem must be chosen so that $A+BK$ is stable. Thus $x(t)$ will approach an invariant subspace $x = \bar{\Pi}w$ where $\bar{\Pi}$ is defined by

$$\bar{\Pi}S = A\bar{\Pi} + P + B(K\bar{\Pi} + E).$$

One can take $\Gamma = K\bar{\Pi} + E$ in this case. Obviously, in order to have $e \to 0$, the solution $\bar{\Pi}$ should also satisfy the condition

$$C\bar{\Pi} - Q = 0.$$

∎

7.2. Error feedback output regulation

THEOREM 7.2. *Suppose (A, B) is stabilizable, the pair*

$$\begin{bmatrix} C & -Q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}$$

is detectable and S is antistable in (7.2). Then the error feedback output regulation is solvable if and only if the linear matrix equations

$$\begin{aligned} \Pi S &= A\Pi + P + B\Gamma \\ 0 &= C\Pi - Q \end{aligned} \tag{7.7}$$

are solved by some Π and Γ.

In other words, under the assumptions, the error feedback problem is solvable if and only if the full information one is solvable.

PROOF
Sufficiency: under the assumption, we can build an observer for (x, w):

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (Cz_1 - Qz_2 - e) + \begin{bmatrix} B \\ 0 \end{bmatrix} u, \tag{7.8}$$

where L_1 and L_2 are chosen such that the error dynamics in (7.9) is stable. Suppose K is such that $A + BK$ is stable. We show

$$u = K(z_1 - \Pi z_2) + \Gamma z_2$$

solves the error feedback problem.

Let $\tilde{z}_1 := z_1 - x$, $\tilde{z}_2 := z_2 - w$ be the tracking errors, then we can rewrite the closed-loop system as

(7.9)
$$\begin{aligned} \dot{x} &= (A + BK)x + Pw - BK\Pi w + B\Gamma w + BK\tilde{z}_1 + B(\Gamma - K\Pi)\tilde{z}_2 \\ \begin{bmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{bmatrix} &= \begin{bmatrix} A + L_1C & P - L_1Q \\ L_2C & S - L_2Q \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} \\ \dot{w} &= Sw \end{aligned}$$

Apparently requirement a) is fulfilled. Then we can easily show that as $t \to \infty$, $(x, \tilde{z})$ tends to the invariant subspace defined by

$$\tilde{z} = 0, \ x = \Pi w.$$

The proof of necessity is similar to the full information case. ∎

7.3. Output regulation and zero dynamics

In this section we discuss the solvability of the Sylvester equation (7.7).

In [**9**] Hautus has shown that the solvability of (7.7) can be characterized by the transmission polynomials of system (7.1) and system (7.2) (where u is considered as the input, and e the output).

PROPOSITION 7.3. *(7.7) is solvable if system (7.1) and system (7.2) have the same transmission polynomials.*

Under some reasonable assumptions we can deduct the following result.

PROPOSITION 7.4. *Suppose (A, B) is stabilizable and (C, A) is detectable, and S is antistable. Then (7.7) is solvable if and only if*

$$\begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}$$

has full row rank for every $s_0 \in \sigma(S)$.

In plain (relatively) language, this implies that system (7.1) is *right-invertible* and its zeros do not coincide with the eigenvalues of S. We use a system that has equal number of inputs and outputs, and has a relative degree $(r_1, \ldots, r_m)$, to illustrate this.

Without loss of generality, we can transform such a system into the normal form:

$$\begin{array}{rcl} \dot{z} & = & Nz + D\xi + P_0 w \\ \dot{\xi}_1^i & = & \xi_2^i + P_1^i w \\ & \vdots & \\ \dot{\xi}_{r_i-1}^i & = & \xi_{r_i}^i + P_{r_i-1}^i w \\ \dot{\xi}_{r_i}^i & = & R_i z + S_i \xi + c_i A^{r_i-1} B u + P_{r_i}^i w \\ y_i & = & \xi_1^i, \; i = 1, \ldots, m \end{array} \tag{7.10}$$

where

$$\xi = (\xi_1^1, \cdots, \xi_{r_1}^1, \cdots, \xi_{r_m}^m)^T.$$

Note that in order to avoid confusion, we have changed a matrix notation for the normal form. As we have studied,

$$\dot{z} = Nz$$

defines the zero dynamics of the system.

In the steady state, $e = 0$ or $y = Qw$ implies that the matrix Q is part of the Π matrix that solves the sylvester equation. Plug in $y_i = \pi_1^i w = Q_i w$ to the normal form, and let $\xi_j^i = \pi_j^i w$, we can obtain iteratively

$$\pi_j^i w = \pi_{j-1}^i S w - P_{j-1}^i w, \; i = 1, ..., m; \; j = 2, ..., r_i.$$

In particular, if $P_j^i = 0, \forall i, j$,

$$\pi_j^i w = y_i^{j-1} = Q_i S^{j-1} w.$$

Thus the only part of Π in (7.7) we need to solve for is corresponding to $z = \Pi_1 w$:

$$\Pi_1 S = N\Pi_1 + D\Pi_2 + P_0, \tag{7.11}$$

where Π_2 has components of π_j^i. It is well known in the literature [**4**] (and mentioned previously) that (7.11) has a unique solution if and only if N and S do not have any common eigenvalue.

Once Π_1 is obtained, once can solve the following equations for Γ:

$$Q_i S^{r_i} w = R_i \Pi_1 w + S_i \Pi_2 w + c_i A^{r_i - 1} B u + P_{r_i}^i w, \ i = 1, ...m,$$

as in this case $u = \Gamma w$.