Mathematical Systems Theory: Advanced Course
Exercise Session 3

1 Transmission zero

Consider a system

$$(\Sigma)\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx, \end{cases}$$

where $x \in R^n$, $u \in R^m$ and $y \in R^p$, B and C are column and row full rank respectively, and (A, B, C) is minimal. A complex number s_0 is called a *transmission zero* if

$$\text{rank} P_\Sigma(s_0) < n + \min(m, p), \;\; P_\Sigma(s) := \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}.$$

How can we compute transmission zeros?

The case where $p = m$: Solve

$$\det P_\Sigma(s) = 0$$

with respect to s.

The case where $p < m$ ($p > m$): Solve

$$\det P_\Sigma(s) P_\Sigma(s)^T = 0 \quad (\det P_\Sigma(s)^T P_\Sigma(s) = 0)$$

with respect to s. Note that $P_\Sigma(s) P_\Sigma(s)^T$ ($P_\Sigma(s)^T P_\Sigma(s)$) is a square matrix.

Note. In MATLAB, the command `tzero.m` computes transmission zeros.

Examples

Square system (A,B,C)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Compute transmission zeros.

Form a system matrix P_Σ:

$$P_\Sigma(s) := \begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} = \begin{bmatrix} s-1 & 0 & -1 & 0 & 0 \\ 0 & s+1 & -2 & 0 & 1 \\ 0 & 0 & s & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \det P_\Sigma(s) &= (s-1)\det\begin{bmatrix} s+1 & -2 & 0 & 1 \\ 0 & s & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} - \underbrace{\det\begin{bmatrix} 0 & -1 & 0 & 0 \\ s+1 & -2 & 0 & 1 \\ 0 & s & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}}_{=0} \\ &= (s-1)\left\{(s+1)\underbrace{\det\begin{bmatrix} s & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=0} + \det\begin{bmatrix} -2 & 0 & 1 \\ s & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\right\} \\ &= s-1 \end{aligned}$$

From $\det P_\Sigma(s) = 0$, we obtain a transmission zero $s = 1$.

Square system (A,B,C,D)

We consider here a system of the form

$$(\bar{\Sigma})\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{cases} \qquad \textit{with}$$

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

For such a system, the Rosenbrock matrix $P_{\bar{\Sigma}}$ is of the form:

$$P_{\bar{\Sigma}}(s) := \begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} s+2 & 0 & -2 & 0 \\ 0 & s & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\det P_{\bar{\Sigma}}(s) = (s+2)\det\begin{bmatrix} s & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} - 1\cdot\det\begin{bmatrix} 0 & -2 & 0 \\ s & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= (s+2)(s+2) - 1\cdot 2(s+2) = (s+2)\cdot s$$

Taking $\det P_{\bar{\Sigma}(s)=0}$ the transmission zeros are $s=0$ and $s=-2$. Notice that these are also eigenvalues of the matrix A, so the transmission zeros and the poles of the system $\bar{\Sigma}$ are equal in this example.

Non-Square system (A,B,C)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The corresponding system matrix is

$$P_{\Sigma}(s) := \begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 & 0 \\ 1 & s+2 & 1 & 1 \\ -1 & -1 & 0 & 0 \end{bmatrix}.$$

To compute transmission zeros, we form $P_{\Sigma}(s)P_{\Sigma}(s)^T$:

$$P_{\Sigma}(s)P_{\Sigma}(s)^T = \begin{bmatrix} s^2+1 & -2 & -s+1 \\ -2 & s^2+4s+7 & -s-3 \\ -s+1 & -s-3 & 2 \end{bmatrix}.$$

The determinant of this matrix is calculated as

$$\det P_{\Sigma}(s)P_{\Sigma}(s)^T = \cdots = 2(s+1)^2.$$

Hence, by setting $\det P_{\Sigma}(s)P_{\Sigma}(s)^T = 0$, we obtain a transmission zero as $s=-1$.

Problem

Compute (both by hand and with computer) transmission zeros of the system with the following (A,B,C).

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

2 High gain control

Here, we will give one example of high gain control.

Example

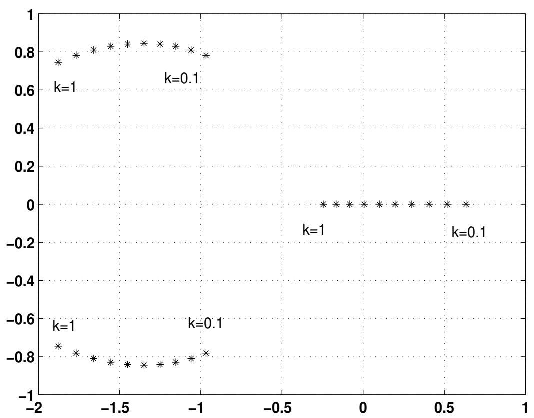
Consider the following system:

$$\begin{cases} \dot{z} &= -\alpha z + \xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \beta z + u \\ y &= \xi_1 \end{cases}$$

In the system, suppose that α is a positive constant but unknown and that β is unknown. From the lecture note, page 38-39, the following control will stabilize the closed-loop system for sufficiently large k:

$$u = -3k\xi_2 - 2k^2\xi_1.$$

The poles of the closed-loop system are shown in the figure below for several k from $k = 0.1$ to $k = 1$. (α and β are set to one.) We can see that large k stabilizes the closed-loop system.



3 Noninteracting control

Given a square system

$$(\Sigma)\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx, \end{cases}$$

where B and C have linearly independent columns rows respectively. Find a control $u = Fx + Gv$ such that

1. the closed-loop system

$$\begin{cases} \dot{x} &= (A+BF)x + BGv \\ y &= Cx \end{cases}$$

has relative degree $(r_1, \cdots, r_m)$, and

2. the i-th output y_i is influenced by only the i-th input v_i.

Solvability condition

The static noninteracting control problem is solvable if and only if the system (Σ) has some relative degree.

How to obtain a solution u?

To obtain a solution u if the problem is solvable, we transform the system (Σ) into a normal form. Then,

$$u = L^{-1}(-Rz - S\xi + v).$$

By this control, we obtain

$$\begin{bmatrix} \dot{\xi}^1_{r_1} \\ \vdots \\ \dot{\xi}^m_{r_m} \end{bmatrix} = \underbrace{\begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}}_{R} z + \underbrace{\begin{bmatrix} S_1 \\ \vdots \\ S_m \end{bmatrix}}_{S} \xi + \underbrace{\begin{bmatrix} c_1 A^{r_1-1} B \\ \vdots \\ c_m A^{r_m-1} B \end{bmatrix}}_{L} u = v,$$

and hence ξ^1_i can be controlled by v_i for each i.

4 Tracking with stability

Consider the same system as above. Find a control $u(t) = Fx(t) + D(t)$ such that

1. the output $y(t)$ tracks asymptotically the reference signal $y_d(t)$
2. the state $x(t)$ is bounded.

Solvability

The tracking problem with stability is solvable if

- the system (Σ) has some relative degree $(r_1, \cdots, r_m)$
- the zero dynamics is asymptotically stable
- for each $i = 1, \ldots, m$,

$$y_d^i, y_d^{i(1)}, \cdots, y_d^{i(r_i-1)}$$

  are bounded.

How to obtain a solution u if the problem is solvable?

$$u(t) = L^{-1}\left(-Rz - S\xi + \begin{bmatrix} y_d^{1(r_1)} \\ \vdots \\ y_d^{m(r_m)} \end{bmatrix} + v(t)\right),$$

where $v(t)$ is chosen so that the closed-loop system becomes asymptotically stable.

Example

Consider the following system which is already in a normal form:

$$\begin{cases} \dot{z} &= \underbrace{-1}_{N} \cdot z + \underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{P} \xi \\ \dot{\xi}_1^1 &= \xi_2^1 \\ \dot{\xi}_2^1 &= \underbrace{1}_{R_1} \cdot z + \underbrace{\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}}_{S_1} \xi + \begin{bmatrix} 2 & 1 \end{bmatrix} u \\ \dot{\xi}_1^2 &= \underbrace{2}_{R_2} \cdot z + \underbrace{\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}}_{S_2} \xi + \begin{bmatrix} 1 & 2 \end{bmatrix} u \\ y_1 &= \xi_1^1 \\ y_2 &= \xi_1^2 \end{cases}$$

In this case,

$$L := \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \; R := \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; S := \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

Assume that the reference signal y_d is given by

$$y_d(t) := \begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix}.$$

We can check that

- The relative degree of this system is $(r_1, r_2) = (2, 1)$.
- The zero dynamics is asymptotically stable (N=-1)
- $y_d^1 = \cos\omega t$, $y_d^{1(r_1-1)} = y_d^{1(1)} = -\sin\omega t$, $y_d^{2(r_2-1)} = y_d^2 = \sin\omega t$ are bounded

By using the control

$$u(t) = L^{-1}\left(-Rz - S\xi + \begin{bmatrix} y_d^{1(r_1)} \\ \vdots \\ y_d^{m(r_m)} \end{bmatrix} + v(t)\right),$$

We get that:

$$\begin{bmatrix} \dot{\xi}_2^1 \\ \dot{\xi}_1^2 \end{bmatrix} = \begin{bmatrix} y_d^{1(2)} + v_1 \\ y_d^{2(1)} + v_2 \end{bmatrix}$$

Defining the tracking errors as in page 44 in the lecture notes,

$$e_j^i = c_i A^{j-1} x - y_d^{i(j-1)} = \xi_j^i - y_d^{i(j-1)} \quad i = 1 \cdots m, \quad j = 1 \cdots r_i$$

we get

$$\begin{bmatrix} e_1^1 \\ e_2^1 \\ e_1^2 \end{bmatrix} = \begin{bmatrix} \xi_1^1 - y_d^{1(0)} \\ \xi_2^1 - y_d^{1(1)} \\ \xi_1^2 - y_d^{2(0)} \end{bmatrix}$$

which implies

$$\begin{bmatrix} \dot{e}_1^1 \\ \dot{e}_2^1 \\ \dot{e}_1^2 \end{bmatrix} = \begin{bmatrix} \dot{\xi}_1^1 - y_d^{1(1)} \\ \dot{\xi}_2^1 - y_d^{1(2)} \\ \dot{\xi}_1^2 - y_d^{2(1)} \end{bmatrix} = \begin{bmatrix} e_2^1 \\ v_1 \\ v_2 \end{bmatrix}$$

we obtain the closed-loop system (Y_d is defined in the lecture notes page 44):

$$\left\{\begin{array}{rcl} \dot{z} & = & Nz + Pe + PY_d \\ \begin{bmatrix} \dot{e}_1^1 \\ \dot{e}_2^1 \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1^1 \\ e_2^1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1 \\ \dot{e}_1^2 & = & v_2 \\ y_1 & = & \xi_1 \\ y_2 & = & \xi_2 \end{array}\right.$$

If we choose v such that the closed-loop system become asymptotically stable, for example

$$\begin{array}{rcl} v_1 & = & \begin{bmatrix} -2 & -3 \end{bmatrix} \begin{bmatrix} e_1^1 \\ e_2^1 \end{bmatrix} \\ v_2 & = & -e_1^2, \end{array}$$

we can check that the tracking problem with stability is solved since

1. $e(t) \to 0$ as $t \to \infty$, since the closed-loop system is asymptotically stable

2. z is bounded since $N = -1$ is a stable matrix (scalar) and ξ is bounded since y_d^1, $y_d^{1(1)}$, $y_d^{2(1)}$ are bounded, so $x(t)$ is bounded.