KTH Matematik

Tentamen i 5B1872 Optimal Control
Saturday March 9 2006 8.00–13.00
Answers and solution sketches

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1. (a) $\lambda(T) = \begin{bmatrix} 1 \\ 1 \\ free \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

(b) There is no feasible solution, hence $= \infty$

(c) $J = 1$

(d) The optimal control for the unconstrained problem, $u(t) = -\dfrac{x(t)}{3-2t}$, satisfies the constraint $|u(t)| \leq 1$ and is thus also optimal for the constrained problem. Indeed, the closed loop state satisfies $|x(t)| \leq 1$.

2. (a) The sequential optimization problem has the form

$$\max \sum_{k=0}^{N-1} u_k \quad \text{subj. to} \quad \begin{cases} x_{k+1} = x_k + \theta(x_k - u_k), \; x_0 \text{ given} \\ 0 \leq u_k \leq x_k \end{cases}$$

The corresponding dynamic programming recursion is

$$\begin{aligned} V_k(x) &= \max_{0 \leq u \leq x} \{u + V_{k+1}(x + \theta(x-u))\} \\ V_N(x) &= 0 \end{aligned}$$

(b) We have

$$\begin{aligned} V_3(x) &= 0 \\ V_2(x) &= \max_{0\leq u \leq x} u = x, && \Rightarrow u_2 = x \\ V_1(x) &= \max_{0\leq u \leq x} \{u + x + \theta(x-u)\} \\ &= \max\{(1+\theta)x, 2x\} = 2x, && \Rightarrow u_1 = x \\ V_0(x) &= \max_{0\leq u \leq x} \{u + 2(x + \theta(x-u))\} \\ &= \max\{(2+\theta)x, 3x\} = 3x, && \Rightarrow u_0 = x \end{aligned}$$

Hence it is optimal to spend all the time. Note, the problem becomes more interesting if the time horizon is longer, i.e. when N is larger.

3. (a) Let us consider the problem in (a) and (b) simultaneulsy. The ARE becomes

$$2ap - p^2 = 0$$

We are generally looking for a postive definite solution. We have

$$p = \begin{cases} 2a, & a > 0 \\ 0, & a < 0 \end{cases}$$

The optimal control becomes

$$u = -px = \begin{cases} -2a, & a > 0 \\ 0, & a < 0 \end{cases}$$

The closed loop system $\dot{x} = (a - p)x = -|a|x$ is stable in both cases. Note that p only is positive semi-definite when $a < 0$, which is a case not covered by the result in the course. However, it is obvious that $u = 0$ is optimal because we only penalize the control and the system converges to zero with $u = 0$.

(b) The open loop system is unstable when $a > 0$ and the control $u = -2ax$ stabilizes the system. The open loop system is stable when $a < 0$ and no control is needed to bring the state to zero. Since the cost function only penalizes the control it is optimal to do nothing.

Note, the case $a = 0$ is not well defined. The solution $u = -\epsilon x$ stabilizes the system but the corresponding cost is

$$\int_0^\infty \epsilon^2 e^{-2\epsilon t} x_0^2 dt = \frac{\epsilon x_0^2}{2}$$

which becomes smaller the smaller ϵ is. However, the limit when $\epsilon = 0$ is not stabilizing, i.e. $u = 0$ does not stabilize the system.

4. The optimal control problem has the formulation

$$\max x_2(1) \quad \text{subj. to} \quad \begin{cases} \dot{x}_1(t) = -x_1(t) + u(t), & x_1(0) = 0 \\ \dot{x}_2(t) = x_1(t), & x_2(0) = 0 \\ 0 \le u \le 1 \end{cases}$$

The Hamiltonian becomes

$$H(x, u, \lambda) = \lambda_1(-x_1 + u) + \lambda_2 x_1$$

From the pointwise optimization we get

$$u = \text{argmax}_{0 \le u \le 1} H(x, u, \lambda) = \begin{cases} 1, & \lambda_1 > 0 \\ 0, & \lambda_1 < 0 \end{cases}$$

We thus expect a switching control law. The adjoint equation becomes

$$\dot{\lambda}_1 = \lambda_1 - \lambda_2$$
$$\dot{\lambda}_2 = 0$$

with terminal condition determined by

$$\lambda(1) - \nabla\Phi(x(1)) \perp S_f$$

where $\Phi(x) = x_2$ and $S_f = \{x : x_1(1) = 0.5\}$. Hence, we get $\lambda_1(1) = free$ and $\lambda_2(1) = 1$. We can now solve the adjoint system, which gives

$$\lambda_1(t) = 1 + (\lambda_1(0) - 1)e^t$$
$$\lambda_2(t) = 1$$

There can be at most one switch in the control function since $\lambda_1(t)$ is a monotonic function. From the problem it is now clear that the control must have the form

$$u(t) = \begin{cases} 1, & 0 \le t < t_s \\ 0, & t_s < t \le 1 \end{cases}$$

We can determine the switching time from the constraint $x(1) = 0.5$. We have $x_1(t_s) = 1 - e^{-t_s}$ and

$$x(1) = e^{-(1-t_s)}(1 - e^{-t}) = 0.5$$

which gives $t_s = \ln(\frac{2+e}{2})$.

5. The optimal control problem is

$$\max \int_0^{t_f} e^{-\alpha t} u(t)p(u(t))dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = -u(t), & x(0) = x_0 \\ t_f \ge 0, x(t_f) = 0 \end{cases}$$

The Hamiltonian is $H(t, x, u, \lambda) = e^{-\alpha t}up(u) - \lambda u$, and the terminal manifold is $S_f(t) = \{0\}$. The adjoint equation is $\dot{\lambda} = -H_x = 0$, which implies $\lambda(t) = const$ (the terminal condition gives no information). The pointwise optimization gives

$$u = \text{argmax}_u e^{-\alpha t}up(u) - \lambda u = \text{argmax}_u e^{-\alpha t}u(1 - u/2) - \lambda u = 1 - \lambda e^{\alpha t}$$

where we used that $u \le 2$ because the cost is zero for $u > 2$, which obviously cannot be optimal. From PMP we have the condition (along the optimal solution)

$$H(t_f, x(t_f), u(t_f), \lambda(t_f)) = -\sum_{k=1}^{p} \nu_k \frac{\partial g_k}{\partial t}(t_f, x(t_f)) - \frac{\partial \phi}{\partial t}(t_f, x(t_f)) = 0$$
$$\Leftrightarrow$$
$$e^{-\alpha t_f}u(t_f)p(u(f_f)) - \lambda u(t_f) = 0$$

which implies $u(t_f) = 0$. Since $u(t_f) = 1 - \lambda e^{\alpha t_f} = 0$ we must have $\lambda = e^{-\alpha t_f}$. The optimal control is $u(t) = 1 - e^{-\alpha(t_f - t)}$, where t_f is determined by the state constraint

$$x(t_f) = x_0 - \int_0^{t_f} u(t)dt = x_0 - t_f - \frac{1}{\alpha}(1 - e^{-\alpha t_f}) = 0 \qquad (1)$$

which is a nonlinear equation in t_f. The optimal value function can now be computed

$$\begin{aligned} V(x, t_f) &= \int_0^{t_f} (u - u^2/2)e^{-\alpha t} dt = 0.5 \int_0^{t_f} (1 - e^{2\alpha(t-t_f)})e^{-\alpha t} dt \\ &= \frac{(1 - e^{-\alpha t_f})^2}{2\alpha} \end{aligned}$$

where once again the terminal time t_f is implicitly defined by equation (1).