KTH Matematik

Exam in 5B1873 Optimal Control
March 8, 2007 at 8.00–13.00
Answers and solution sketches

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1. (*a*)

$$\min x(t_f) \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = v\cos(\theta(t)) + c, & x(0) = 0, \\ \dot{y}(t) = v\sin(\theta(t)), & y(0) = 0,\ y(t_f) = y_f \\ t_f > 0 \end{cases}$$

(*b*) $H(x, y, \theta, \lambda) = \lambda_1(v\cos(\theta) + c) + \lambda_2 v\sin(\theta)$. Pointwise minimization gives

$$\min(\lambda_1(v\cos(\theta) + c) + \lambda_2 v\sin(\theta)) = \lambda_1 c - v\sqrt{\lambda_1^2 + \lambda_2^2}$$

with optimal direction determined by

$$\begin{bmatrix} \cos(\theta^*) \\ \sin(\theta^*) \end{bmatrix} = -\frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2}} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

The adjoint equation is defined by

$$\dot{\lambda}_1(t) = 0, \quad \lambda_1(t) = \lambda_1^0$$
$$\dot{\lambda}_2(t) = 0, \quad \lambda_2(t) = \lambda_2^0$$

so the direction must be constant.

(*c*) The boundary conditions for the adjoint variable satisfies

$$\lambda_1(t_f) = 1$$
$$\lambda_2(t_f) = \nu,\ \nu \in R$$

Unfortunately, this does not provide sufficient information in order to continue. Instead we consider the terminal condition

$$x(t_f) = (v\cos(\theta^*) + c)t_f = x_f$$
$$y(t_f) = v\sin(\theta^*)t_f = y_f$$

which implies

$$x_f(\theta^*) = \frac{v\cos(\theta^*) + c}{v\sin(\theta^*)} y_f$$

We want to minimize $x_f(\theta^*)$ and therefore solve for the stationary point

$$x_f'(\theta^*) = \frac{-v + c\cos(\theta^*)}{v\sin^2(\theta^*)} y_f = 0$$

which is the case when

$$\theta^* = \cos^{-1}(-v/c)$$

2. (*a*) We have $H(x,u,\lambda) = 2\lambda u$. This gives $\tilde{\mu}(x,\lambda) = -\text{sign}(\lambda)$ and thus the HJBE becomes

$$\begin{cases} -V_t = H(x, \tilde{\mu}(x, V_x), V_x) \\ V(T,x) = \phi(x) \end{cases} \quad \Leftrightarrow \quad \begin{cases} -V_t = -2V_x \text{sign}(V_x) \\ V(1,x) = x^2 \end{cases}$$

Hence, alternative (*a*) is correct.

(*b*) The second attempt is using the dynamic programming equation correctly. In the first attempt the two time segments are treated independently, which is a violation of the dynamic programming equation and the principle of optimality.

3. Both problems have the same solution. Here we only give the proof of part (*b*), which is a bit harder that (*a*). The Hamiltonian is

$$H(x,u,\lambda) = u_1^2 + u_2^2 + \lambda_1 u_1 + \lambda_2 u_2$$

Pointwise minimization gives

$$u^* = \mu(\lambda) = \begin{bmatrix} -\lambda_1/2 \\ -\lambda_2/2 \end{bmatrix}$$

The adjoint equation is

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1^0 \\ \lambda_2^0 \end{bmatrix}$$

The boundary conditions for the adjoint variable reduces to

$$\lambda(0) = \begin{bmatrix} 1 \\ 2_2(0) \end{bmatrix} \nu_1, \qquad \lambda(2) = \begin{bmatrix} -1 \\ 2x_2(2) \end{bmatrix} \nu_2$$

where $\nu_1\nu_2 \in \mathbf{R}$. Since $\lambda(t) = \lambda^0$ (constant) we must have $\nu_2 = -\nu_1$. Clearly, this requires that $x_2(0) = x_2(2) = 0$, which gives the control

$$u = \begin{bmatrix} \nu_1 \\ 0 \end{bmatrix}$$

The solution becomes

$$x(t) = \begin{bmatrix} \nu_1 t \\ 0 \end{bmatrix}$$

In order for $x(2) \in S_1$ we must have $\nu_2 = 0.5$.

4. We may use the dynamic programming algorithm with

$$V(k,x) = \min_{|u|\le 1} \{|u| + V(k+1, x+u)\}$$
$$V(2,x) = 2|x|$$

At $k = 1$ we get

$$V(1,x) = \min_{|u|\le 1} \{|u| + 2|x+u|\} = \begin{cases} 1 + 2|x-1|, & x > 1 \\ |x|, & |x| \le 1 \\ 1 + 2|x+1|, & x < -1 \end{cases}$$

with corresponding controls

$$u_1^* = \begin{cases} -1, & x > 1 \\ -x, & |x| \le 1 \\ 1, & |x| < -1 \end{cases}$$

At $k = 0$ we have three cases

(a) If $|x+u| \le 1$ (possible when $|x| \le 2$) then

$$V(0,x) = \min_{|u|\le 1} \{|u| + |x+u|\} = \begin{cases} 1 + 2|x-1|, & 1 < x \le 2 \\ |x|, & |x| \le 1 \\ 1 + |x+1|, & -2 \le x < -1 \end{cases}$$

with corresponding controls

$$u_0^* = \begin{cases} -1, & 1 < x \le 2 \\ -y, & 0 \le y \le x \le 1 \\ -y, & -1 \le x \le y \le 0 \\ 1, & -2 \le x < -1 \end{cases}$$

(b) If $x + u > 1$ (possible when $x > 2$) then

$$V(0,x) = \min_{|u|\le 1} \{|u| + 1 + 2|x+u-1|\} = 2 + 2|x-2|$$

and $u_0^* = -1$

(c) If $x + u < -1$ (possible when $x < -2$) then

$$V(0,x) = \min_{|u|\le 1} \{|u| + 1 + 2|x+u+1|\} = 2 + 2|x+2|$$

and $u_0^* = 1$.

Hence, one possible explicit MPC is (with $y = x$ above)

$$u_{t|t} = \mu(x_{t|t}) = \begin{cases} -1, & x > 1 \\ -x, & -1 < x < 1 \\ 1, & x < -1 \end{cases}$$

5. (*a*) Only $u = -(1+\sqrt{2})$ gives a closed loop system that converges to zero. This is the optimal solution.

(*b*) The Hamilton-Jacobi-Bellman equation becomes

$$0 = \min_u \left\{ x^T x + u^2 + V_x(x)^T (Ax + Bu) \right\}$$

It is easy to see that $V(x) = x^T P x$ is a solution if P solves the ARE

$$A^P + PA + I = PBB^T P$$

The optimal control is $u = -B^T P x$. There are many solutions to the ARE but only the positive definite solution gives a stable closed loop system, i.e. only when $P > 0$ will the closed loop solution converge to zero. We know from the lecture notes that there always exists a positive definite solution to the ARE under the stated conditions.

(*c*) The ARE $A^T P + PA + I - PBB^T P = 0$ becomes

$$\begin{bmatrix} -p_{12}^2 + 1 & -p_{12}p_{22} + p_{11} \\ -p_{12}p_{22} + p_{11} & 2p_{12} - p_{22}^2 + 1 \end{bmatrix} = 0$$

which implies

$$P_{12} = \pm 1$$
$$P_{22} = P_{11} = \pm\sqrt{1 \pm 2}$$

The positive definite solution to the ARE is

$$P = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}$$

The optimal state feedback is $u = -R^{-1}B^T P x = -\begin{bmatrix} 1 & \sqrt{3} \end{bmatrix} x$ and the closed loop system matrix becomes

$$A - BB^T P = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{3} \end{bmatrix}$$

which is a stable matrix. The optimal cost is

$$V(x_0) = x_0^T P x_0 = 2(1 + \sqrt{3})$$