Exercise 14.6

We have

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) & \frac{\partial f}{\partial x_2}(x) \end{bmatrix}$$

$$= \begin{bmatrix} -2x_1(x_2-3x_1^2)+(x_2-x_1^2)(-6x_1) & x_2-3x_1^2+x_2-x_1^2 \end{bmatrix}$$

and so

$$\nabla f(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Also,

$$F(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) \end{bmatrix}$$

$$= \begin{bmatrix} \begin{array}{l} -2(x_2-3x_1^2)-2x_1(-6x_1) \\ -2x_1(-6x_1)+(x_2-x_1^2)(-6) \end{array} & -6x_1-2x_1 \\ -6x_1-2x_1 & 2 \end{bmatrix}$$

and so

$$F(0) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \geq 0.$$

(because $x^T F(0) x = 2x_2^2 \geq 0$)

Exercise 14.9

We have

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}(x) \quad \frac{\partial f}{\partial x_2}(x) \right] = \left[4x_1^3 \quad 4x_2^3 \right]$$

and so $\nabla f(0) = [0 \quad 0]$

Also

$$F(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) \end{bmatrix} = \begin{bmatrix} 12x_1^2 & 0 \\ 0 & 12x_2^2 \end{bmatrix}$$

and so

$$F(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$$

F(0) is not positive definite since for example

$$x^T F(0) x = 0 \quad \forall x \in \mathbb{R}^2$$

Exercise 14.10

We have $\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \frac{\partial f(x)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 3x_1^2 - 12x_2 & -12x_1 + 24x_2^2 \end{bmatrix}$

and so $\nabla f(x)\Big|_{x=\hat{x}=\begin{bmatrix}2\\1\end{bmatrix}} = \begin{bmatrix} 3\cdot 4 - 12\cdot 1 & -12\cdot 2 + 24\cdot(1)^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

Moreover, $F(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 6x_1 & -12 \\ -12 & 48x_2 \end{bmatrix}$

and so $F(x)\Big|_{x=\hat{x}=\begin{bmatrix}2\\1\end{bmatrix}} = \begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix} = 12 \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}_{H}$.

We have

$$d^T H d = d_1^2 - 2d_1d_2 + 2d_2^2 = (d_1 - d_2)^2 + d_2^2 \geq 0$$

$$= 0 \quad \text{iff} \quad d_1 = d_2 = 0.$$

So $\hat{x}$ is a local minimizer.

Exercise 14.11

We first calculate candidates for local minimizers

$$\nabla g(x) = [4x_1^3 - 12x_2 \quad -12x_1 + 4x_2^3]$$

Thus $\nabla g(x) = 0$ iff $\left\{\begin{array}{l} 4x_1^3 = 12x_2 \\ 4x_2^3 = 12x_1 \end{array}\right\}$, i.e.,

iff $(x_1^3 = 3x_2$ and $x_1 = \frac{1}{3}x_2^3)$ (*)

If (*) holds, then $\frac{1}{3^3}x_2^9 = 3x_2$ i.e., $x_2(x_2^8 - 3^4) = 0$.

So $x_2 = 0$ or $x_2^2 = 3$ i.e., $x_2 \in \{0, \sqrt{3}, -\sqrt{3}\}$

If $x_2 = 0$, $x_1 = 0$; if $x_2 = \sqrt{3}$, $x_1 = \sqrt{3}$; if $x_2 = -\sqrt{3}$, $x_1 = -\sqrt{3}$

So if (*) holds, then $x \in \{(0,0), (\sqrt{3}, \sqrt{3}), (-\sqrt{3}, -\sqrt{3})\}$.

Conversely, if $x \in \{(0,0), (\sqrt{3},\sqrt{3}), (-\sqrt{3}, -\sqrt{3})\}$, then (*) holds.

We have $g(0,0) = 0$, while

$$g(\sqrt{3}, \sqrt{3}) = g(-\sqrt{3}, -\sqrt{3}) = 18 - 12\cdot 3 = -18 < g(0,0)$$

Since we are interested in global minimizers, we discard $(0,0)$. Also, the Hessian $G(x)$ of g at x is given by

$$G(x) = \begin{bmatrix} 12x_1^2 & -12 \\ -12 & 12x_2^2 \end{bmatrix} = 12 \begin{bmatrix} x_1^2 & -1 \\ -1 & x_2^2 \end{bmatrix}.$$

Thus $G(\sqrt{3}, \sqrt{3}) = G(-\sqrt{3}, -\sqrt{3}) = 12 \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$, which is positive definite.

(For a suitable elementary matrix E_1,
$E_1 \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} E_1^T = \begin{bmatrix} 3 & 0 \\ 0 & 3 - 1/3 \end{bmatrix}$ and $3 > 0$, $3 - 1/3 > 0$.)

So $(\sqrt{3}, \sqrt{3})$, $(-\sqrt{3}, -\sqrt{3})$ are both local minimizers with the value of g being the same at these points.

We have

$$g(x) = x_1^4 + x_2^4 - 12 x_1 x_2 \geq 2\left(x_1^2 + x_2^2\right)^2 - 12\left(\frac{x_1^2 + x_2^2}{2}\right)$$
$$= 2\,\|x\|_2^4 - 6\,\|x\|_2^2$$
$$\longrightarrow \infty \quad \text{as} \quad \|x\|_2 \longrightarrow \infty,$$

Thus $\exists R > 0$ s.t. $\forall \|x\|_2 \geq R$, $g(x) > -18 = g(\sqrt{3}, \sqrt{3})$
$= g(-\sqrt{3}, -\sqrt{3})$.

In the compact set
$$K := \{ x \in \mathbb{R}^2 \mid \|x\|_2 \leq R \},$$
$g|_K$ has a global minimizer. But then these have to be the points $(\sqrt{3}, \sqrt{3})$, $(-\sqrt{3}, -\sqrt{3})$. So they now serve as global minimizers for g in $\mathbb{R}^2$
($g(\sqrt{3}, \sqrt{3}) = g(-\sqrt{3}, -\sqrt{3}) \leq g(x)$ for $x \in K$ as well as
$g(\sqrt{3}, \sqrt{3}) = g(-\sqrt{3}, -\sqrt{3}) = -18 \leq g(x)$ for $x \notin K$.).