# SF1811 Optimization

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Optimization and Systems Theory
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1. Course overview and application examples
2. Course information and course logistics

# What is this course about?

Short answer:

- The art of doing things as good as possible – using mathematical models and methods.

Slightly extended answer:

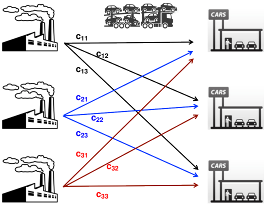
- Mathematical models for optimization (i.e., useful mathematical formulations of optimization problems for various applications).
- Methods for solving mathematically formulated optimization problems.
- Theory for optimization.

# Main parts of the course

- Linear optimization.
- Optimization of flows in networks.
- Quadratic optimization.
- Unconstrained nonlinear optimization.
- Nonlinear optimization subject to given constraints.

Constantly present in the course: Linear algebra.

# A transportation problem




Given:

- $m$ “plants” (origins, sources) with given supplies $s_1, \ldots, s_m$ units.
- $n$ “customers” (destinations, sinks) with given demands $d_1, \ldots, d_n$ units.
- $c_{ij}$ = cost per transported unit from plant $i$ to customer $j$.

We seek a plan that minimizes the total transportation cost.

Question to be answered for each pair $(i, j)$:

- How many units are transported from plant $i$ to customer $j$?

# A transportation problem, cont.

Introduce the optimization variables

$x_{ij}$ = number of units transported from plant i to customer j.

Then the mathematical model becomes:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij} \leq s_i, \quad i = 1, \ldots, m \\
& \sum_{i=1}^{m} x_{ij} = d_i, \quad j = 1, \ldots, n \\
& x_{ij} \geq 0, \quad \text{for all } i \text{ and } j.
\end{aligned}
$$

Property: If all the $s_i$ and $d_j$ are integers then an optimal solution with all the $x_{ij}$ being integers will be found!

# An assignment problem (tilldelningsproblem)

Invited to a dinner are $n$ ladies and $n$ gentlemen, say $n = 30$.

We search for an assignment of gentlemens to ladies (or vice versa) which maximizes the total pleasure at the dinner table.

For each pair $(i, j)$, let $p_{ij} \in \{1, 2, \ldots, 10\}$ be the grade of "pleasure contribution" if gentleman $i$ sits with lady $j$ at the table.

Even if all these $p_{ij}$ have been estimated (or obtained from the guests) the optimization problem seems hopeless if $n = 30$.

The number of possible solutions is $30! > 10^{32}$ the age of the universe in microseconds.

But it can be solved!

# An assignment problem, cont.

Introduce binary optimization variables $x_{ij}$ with the interpretation $x_{ij} = 1$ if gentleman $i$ sits with lady $j$, $x_{ij} = 0$ otherwise.

Then the mathematical model becomes:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}x_{ij} \\ \text{subject to} & \sum_{j=1}^{n} x_{ij} \leq 1, \quad i = 1,\dots,m, \\ & \sum_{i=1}^{m} x_{ij} = 1, \quad j = 1,\dots,n, \\ & x_{ij} \in \{0,1\}, \text{ for all } i \text{ and } j. \end{array}$$

The Property implies that $x_{ij} \in \{0,1\}$ can be replaced by $x_{ij} \geq 0$. An optimal solution with all $x_{ij} \in \{0,1\}$ will still be found!

# Portfolio optimization (Markowitz)

Given:
$K$ = a given capital to invest during the coming year,
$n$ = number of different assets (e.g., stocks/bonds) to invest in,
$r_j$ = the return of asset j (a random variable!),
$\bar{r}_j = E[r_j]$ = the expected return of asset $j$,
$c_{jj} = \sigma_{jj}^2 = \text{Var}(r_j)$ = the variance of the return of asset j.
$c_{ij} = \text{Cov}(r_i, r_j)$ = the covariance of the returns of assets i and j.

Introduce the variable vector $x = (x_1, \cdots, x_n)^T$, where
$x_j$ = the amount we choose to invest in asset $j$,
i.e., $0 \leq x_j \leq K$ and $\sum_{j=1}^n x_j = K$.

Then the return of our portfolio is the random variable $\sum_{j=1}^n x_j r_j$.

# Portfolio optimization, cont.

The expected return of our portfolio will be

$$\mu(x) = E[\sum_{j=1}^{n} x_j r_j] = \sum_{j=1}^{n} x_j E[r_j] = \sum_{j=1}^{n} x_j \bar{r}_j = \bar{r}^T x,$$

while the variance of the return of our portfolio will be

$$\sigma^2(x) = E[(\sum_{j=1}^{n} x_j r_j - \mu(x))^2] = E[(\sum_{j=1}^{n} x_j (r_j - \bar{r}_j))^2]$$

$$= E[(\sum_{i=1}^{n} x_i (r_i - \bar{r}_i))(\sum_{j=1}^{n} x_j (r_j - \bar{r}_j))] = E[\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j (r_i - \bar{r}_i)(r_j - \bar{r}_j)]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j E[(r_i - \bar{r}_i)(r_j - \bar{r}_j)] = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j c_{ij} = x^T C x$$

where $C$ is the covariance matrix.

# Portfolio optimization, cont.

Two reasonable optimization problems based on the above assumptions:

$$\begin{array}{ll} \text{minimize} & x^T C x \\ \text{subject to} & \bar{r}^T x \geq \alpha \\ & e^T x = K, \quad (e = (1, \ldots, 1)^T) \\ & x \geq 0, \end{array}$$

for one or several different values on the right hand side $\alpha$.

$$\begin{array}{ll} \text{maximize} & \bar{r}^T x - \rho x^T C x \\ \text{subject to} & e^T x = K, \quad (e = (1, \ldots, 1)^T) \\ & x \geq 0, \end{array}$$

for one or several different values on the “price” $\rho$ on risk.

# Nonlinear optimization subject to given constraints

Variable vector $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$. Considered problem:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0, \quad i = 1, \dots, m_1, \\ & h_k(x) = 0, \quad k = 1, \dots, m_2, \end{aligned}$$

where $f(x)$, $g_i(x)$ and $h_k(x)$ are differentiable functions.

This is the most general problem considered in this course.

# Optimal sizing of a truss structure (sv. fackverk)

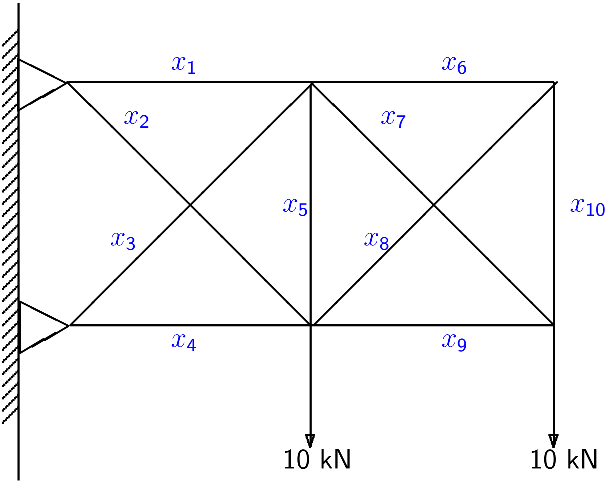



Figure 1: $x_j$ = the cross section area of the $j$:th bar.

# Displacements of the nodes

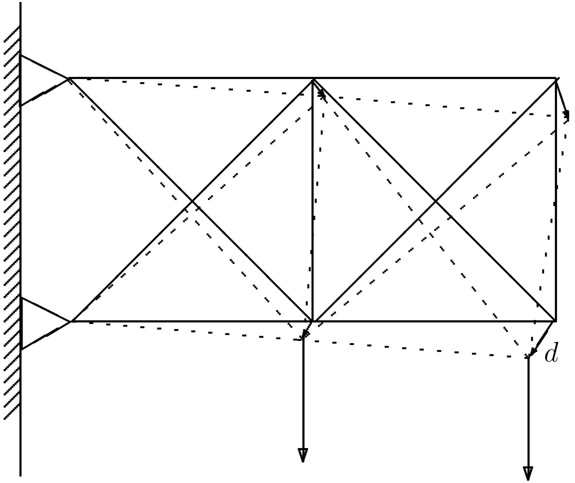



Figure 2: $d = d(x_1, \ldots, x_{10}) = d(x)$

# Possible optimization problem

Minimize the weight subject to a stiffness constraint:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{10} L_j x_j \\
\text{subject to} \quad & d(x) \le d^{\max}, \\
& x_j \ge 0, \quad j = 1, \ldots, 10,
\end{aligned}
$$

where $x = (x_1, \ldots, x_{10})^T \in \mathbb{R}^{10}$ is the variable vector,
$L_j$ is the given length of the $j$th bar
and $d^{\max}$ is a given constant.

# First optimization course

Sets the mathematical foundations for further studies in:

- Optimal control (problems with dynamics)
- Radiation therapy, Medical imaging
- Portfolio optimization and prediction
- Optimal scheduling and planning
- Network problems, Consensus problems
- Data analysis, Machine learning, Neural networks

# Course information

Course information available at the course homepage:

`https://www.math.kth.se/optsyst/grundutbildning/kurser/SF1811/`

(Address available at Social and at my homepage)

# Teachers

- Main teacher:
  Johan Karlsson (Email: johan.karlsson@math.kth.se)
- Teaching assistants:
  - Tove Odland, (Email: odland@kth.se)
    In English, in the first of the two scheduled rooms (L51, etc.)
  - David Ek, (Email: daviek@kth.se)
    In Swedish, in the second of the two scheduled rooms (L52, etc.)

Teaching assistants' page:

`https://people.kth.se/~daviek/SF1811optHT17.html`

(Available from course homepage)

# Course Material

- “Optimization” by Amol Sasane and Krister Svanberg, which you can buy at the KTH bookstore.
- Additional exercises: "Exercises in Optimization" available at course homepage
- Recommended reading: “Linear and Nonlinear Optimization,” second edition, by Griva, Nash and Sofer. This is course literature on in the follow up courses SF2812 and SF2822.

# Homeworks and exam

**Homeworks:**

- Two optional homework sets
- Each set gives up to 2 bonus points at the exam

The **final exam** takes place Wednesday 2018-01-10, 14-19.

**You must register for the exam during 20 nov - 18 dec 2017. Use "My Pages".**

- Total credit = exam score + homework score.
- The maximum exam score = 50. Maximum bonus from the homework sets = 4.
- You are guaranteed to pass if you get 25 credits.
- The tasks are written in English, but you may write your answers in either English or Swedish.