Exercise 20.15

Let $g_1(x) := -x_1$

$g_2(x) := -x_2$

$g_3(x) := x_2 - (x_1 - 1)^2.$

Then the feasible set is $\mathcal{S} = \{x \in \mathbb{R}^2 : g_i(x) \leq 0, i = 1,2,3\}.$

At $x = (1,0)$, $g_1(x) = -1 < 0$,

$g_2(x) = -0 = 0$,

$g_3(x) = 0 - (1-1)^2 = 0.$

So the active index set $I_a(x) = \{2,3\}$.

Since $g_i(x) \leq 0$ for $i = 1,2,3$, $x \in \mathcal{S}$, i.e., $(1,0)$ is feasible.

We have

$$\nabla g_2(x) = [0 \quad -1],$$

$$\nabla g_3(x) = [-2(x_1 - 1) \quad 1]\Big|_{(1,0)} = [0 \quad 1].$$

Since $1 \cdot \nabla g_2(x) + 1 \cdot \nabla g_3(x) = 0$, $\nu_2 := 1 \geqslant 0$, $\nu_3 := 1 \geqslant 0$,

and $\nu_2 + \nu_3 = 1 + 1 = 2 > 0$,

it follows that x is not a regular point.

Exercise 20.16

Let $g_1(x,y) := x+y-1$

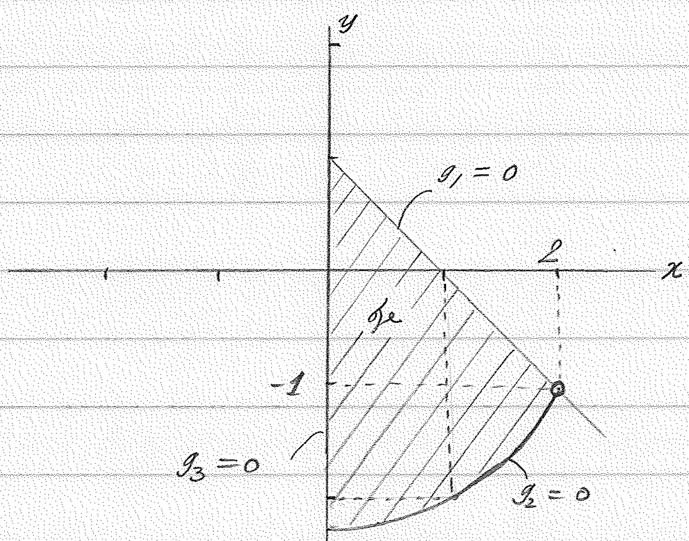
$g_2(x,y) := x^2+y^2-5$

$g_3(x,y) := -x$.

Then the region is given by

$$\mathcal{F} := \{ (x,y) \in \mathbb{R}^2 : g_i(x,y) \le 0 ,\ i=1,2,3\},$$

and is shown below:



The active index set at $\hat{x} = (2,-1)$ is $I_a(\hat{x}) = \{1, 2\}$.

We have $\nabla g_1(\hat{x}) = [1 \quad 1]$,

$\nabla g_2(\hat{x}) = [2x \quad 2y]\big|_{(2,-1)} = [4 \quad -2]$.

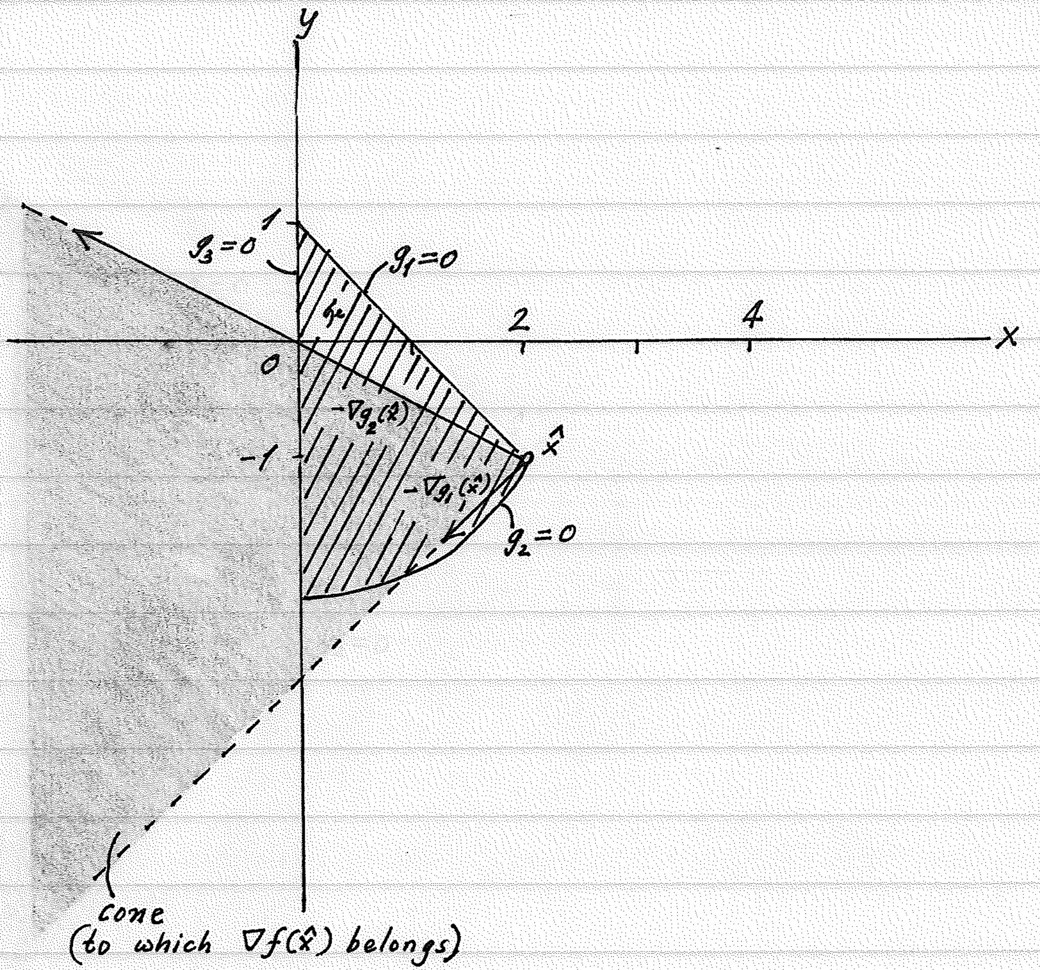
$\nabla g_1(\hat{x})$, $\nabla g_2(\hat{x})$ are linearly independent, and so $\hat{x}$ is a regular point.

Since $\hat{x}$ is an optimal solution, the KKT-conditions should be satisfied. In particular, $\exists\ \hat{y}_1, \hat{y}_2, \hat{y}_3 \geq 0$ such that $\nabla f(\hat{x}) + \hat{y}_1 [1 \quad 1] + \hat{y}_2 [4 \quad -2] + \hat{y}_3 [-1 \quad 0] = 0$. But $y_3 = 0$ since $\hat{y}_3\, g_3(\hat{x}) = 0$ and $g_3(\hat{x}) = -2 \neq 0$.

Hence $\nabla f(\hat{x})$ belongs to the "cone" generated by $-\nabla g_1(\hat{x})$ and $-\nabla g_2(\hat{x})$, that is,

$$\nabla f(\hat{x}) = \hat{y}_1 (-[1 \quad 1]) + \hat{y}_2 (-[4 \quad -2]).$$

This is shown in the following figure:



y
x
1
0
2
4
-1
$g_3=0$
$g_1=0$
$g_2=0$
$\hat{x}$
$-\nabla g_2(\hat{x})$
$-\nabla g_1(\hat{x})$
cone
(to which $\nabla f(\hat{x})$ belongs)

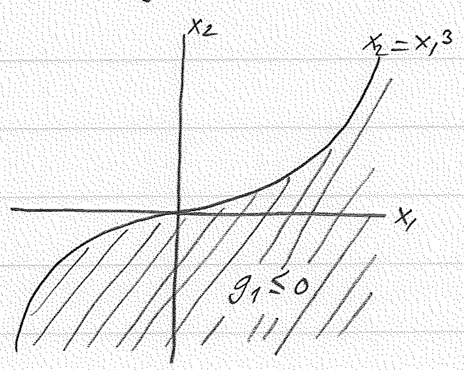
Exercise 20.17.

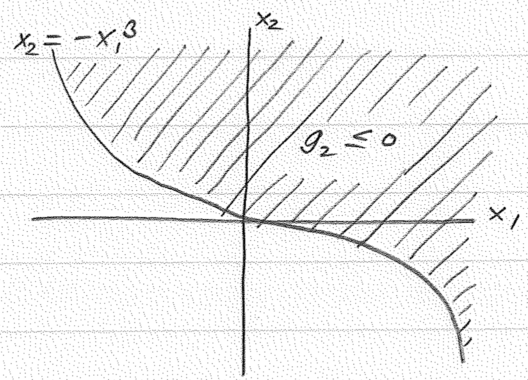
(1) The problem is

$$(NP): \begin{cases} \text{minimize} & f(x_1, x_2) \\ \text{subject to} & g_1(x_1, x_2) \le 0, \\ & g_2(x_1, x_2) \le 0, \end{cases}$$

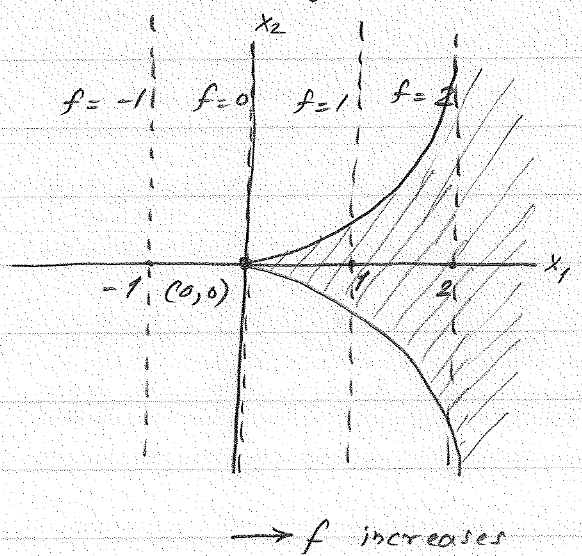
where
$$f(x_1, x_2) := x_1,$$
$$g_1(x_1, x_2) := x_2 - x_1^3,$$
$$g_2(x_1, x_2) := -x_2 - x_1^3.$$

The regions $g_1 \le 0$ and $g_2 \le 0$ look like this:





Thus the feasible region looks like this:



The level curves of f are shown by dotted lines. From the figure above we see that $(0, 0)$ is a global minimizer.

(2) The KKT - conditions are given by

(KKT-1) $\nabla f(x) + y^T \nabla g(x) = 0$.

In our case $\nabla f(x) = [1 \quad 0]$

$$\nabla g(x) = \begin{bmatrix} \nabla g_1(x) \\ \nabla g_2(x) \end{bmatrix} = \begin{bmatrix} -3x_1^2 & +1 \\ -3x_1^2 & -1 \end{bmatrix}.$$

So we have

$$[1 \quad 0] + [y_1 \quad y_2] \begin{bmatrix} -3x_1^2 & 1 \\ -3x_1^2 & -1 \end{bmatrix} = [0 \quad 0]$$

i.e, $\begin{cases} 1 - 3x_1^2 y_1 - 3x_1^2 y_2 = 0 \\ y_1 - y_2 = 0. \end{cases}$

(KKT-2) $g(x) \leq 0$.

In our case: $\begin{cases} x_2 - x_1^3 \leq 0 \\ -x_2 - x_1^3 \leq 0. \end{cases}$

(KKT-3) $y \geqslant 0$.

In our case: $\begin{cases} y_1 \geqslant 0 \\ y_2 \geqslant 0. \end{cases}$

(KKT-4) $y_i g_i(x) = 0$ for $i = 1, \dots, m$.

In our case: $\begin{cases} y_1(x_2 - x_1^3) = 0 \\ y_2(-x_2 - x_1^3) = 0. \end{cases}$

For the point $(0,0)$, (KKT-2) and (KKT-4) are clearly satisfied. However (KKT-1) can never be satisfied since $1 - 3x_1^2 y_1 - 3x_1^2 y_2 \big|_{(x_1, x_2) = (0,0)} = 1 \neq 0$.

So there is no $y \in \mathbb{R}^2$ for which (KKT-1) - (KKT-4) are satisfied at $(0,0)$.

We know that the KKT - conditions are necessary for a local (and hence global) optimal solution x

under the condition that x is a regular point.

However, we have: $I_a(0,0) = \{1,2\}$ since $g_1(0,0) = 0$ and $g_2(0,0) = 0$.

Moreover, $\nabla g_1(0) = [0 \quad 1]$,

$\nabla g_2(0) = [0 \quad -1]$,

and we have with $v_1 := 1 \geq 0$, $v_2 := 1 \geq 0$

that $\sum_{i \in I_a(0,0)} v_i = 1 + 1 = 2 > 0$ and

$$\sum_{i \in I_a(0,0)} v_i \nabla g_i(0,0) = v_1 \cdot \nabla g_1(0) + v_2 \cdot \nabla g_2(0) = 1 \cdot [0 \quad 1] + 1 \cdot [0 \quad -1] = 0.$$

So $(0,0)$ is <u>not</u> a regular point.