Exercise 23.2

Denote by $S(X)$ the intersection of all subspaces that contain X. (The set of subspaces containing X is not empty, since for example $\mathbb{R}^n$ contains X.)

S1 $0 \in S(X)$, since every subspace contains it.

S2 Suppose $v_1, v_2 \in S(X)$. Then if S' is a subspace containing X, $v_1, v_2 \in S(X) \subset S'$. Since S' is a subspace, $v_1 + v_2 \in S'$. So $v_1 + v_2$ belongs to every subspace containing X. Hence $v_1 + v_2 \in S(X)$

S3 Let $\alpha \in \mathbb{R}$ and $v \in S(X)$. If S' is a subspace containing X, then $v \in S(X) \subset S'$. Since S' is a subspace, $\alpha . x \in S'$. So $\alpha . x$ belongs to every subspace containing X. Hence $\alpha \cdot x \in S(X)$.

So $S(X)$ is a subspace.

Clearly $S(X)$ contains X since it is the intersection of all subspaces containing X.

Suppose S' is a subspace containing X. Then $S(X) \subset S'$. So $S(X)$ is the smallest subspace of $\mathbb{R}^n$ that contains X.

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span $\emptyset = \{0\}$: Since $\emptyset \subset \{0\}$, we have span $\emptyset \subset \{0\}$. But also $0 \in$ span $\emptyset$, and so span $\emptyset = \{0\}$.

$X \subset$ span X, and so $v_1, \dots, v_k \in$ span X. Since span X is a subspace, also $\alpha_1 \cdot v_1 + \dots + \alpha_k \cdot v_k \in$ span X for all scalars $\alpha_1, \dots, \alpha_k \in \mathbb{R}$. So $\{\alpha_1 v_1 + \dots + \alpha_k \cdot v_k : \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$ is contained in span X. It is clear that $\{\alpha_1 . v_1 + \dots + \alpha_k . v_k : \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$ containing $v_1, \dots, v_k$. So span $X \subset \{\alpha_1 v_1 + \dots + \alpha_k . v_k : \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$

This completes the proof.

Exercise 23.3

S1. $0^T = 0$, and so the zero matrix $O \in \mathbb{R}^{n\times n}$ belongs to S.

S2. Suppose $A, B \in S$. Then $A^T = A$, $B^T = B$.
Then $(A+B)^T = A^T + B^T = A + B$, and so $A+B \in S$.

S3. If $A \in S$ and $\alpha \in \mathbb{R}$, then $(\alpha A)^T = \alpha A^T = \alpha A$ and so $\alpha A \in S$.

So S is a subspace of $\mathbb{R}^{n\times n}$.

Exercise 23.4

span $\emptyset = \{0\}$ (Exercise 23.2), and $\emptyset$ is linearly independent.

Thus it forms a basis for $\{0\}$

Exercise 23.5

Let E_{ij} be the $n \times n$ matrix with 1 in the ith row and jth column, and all other entries equal to 0.

Let $B = \{ E_{ij} + E_{ji} : 1 \leq i < j \leq n \} \cup \{ E_{ii} : 1 \leq i \leq n \}$.

We claim that B forms a basis for S.

Suppose that $A \in S$. Since $A = A^T$, A has the form

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{aligned} &= a_{11} E_{11} + \cdots + a_{nn} E_{nn} + a_{12}(E_{12} + E_{21}) + \cdots + a_{1n}(E_{1n} + E_{n1}) \\ &\quad + a_{23}(E_{23} + E_{32}) + \cdots + a_{2n}(E_{2n} + E_{n2}) \\ &\quad + a_{34}(E_{34} + E_{43}) + \cdots + a_{3n}(E_{3n} + E_{n3}) \\ &\quad + \cdots \\ &\quad + a_{(n-1),n}(E_{(n-1),n} + E_{n,(n-1)}). \end{aligned}$$

$\in$ span B.

Conversely, span B is clearly contained in S.

B is linearly independent since if

$$\alpha_{11} E_{11} + \cdots + \alpha_{nn} E_{nn} + \sum_{1 \leq i < j \leq n} \alpha_{ij} (E_{ij} + E_{ji}) = 0,$$

then

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{12} & \ddots & & \vdots \\ \vdots & & & \\ \alpha_{1n} & \cdots & & \alpha_{nn} \end{bmatrix} = 0,$$

and so $\alpha_{11} = \cdots = \alpha_{nn} = \alpha_{ij} = 0$ for all $i < j$.

Thus B forms a basis for S.

Exercise 23.7

dim $\{0\}$ = cardinality $\emptyset = 0$.

Exercise 23.8.

$$\dim S = \text{cardinality}\left(\{E_{ij}+E_{ji} : 1\le i<j\le n\} \cup \{E_{ii} : 1\le i\le n\}\right)$$

(see the solution to Exercise 23.5)

$$= \sum_{i=1}^{n} n-i + \sum_{i=1}^{n} 1$$

$$= n^2 - \frac{n(n+1)}{2} + n$$

$$= \frac{n(n+1)}{2}$$

Exercise 23.9

$S_1 \cap S_2$, being a subspace of S_1, is finite-dimensional as well. Let $\{v_1, \dots, v_k\}$ be a basis for $S_1 \cap S_2$. In particular, it is independent, and can be extended to a basis for S_1 by adding extra vectors, say, $u_1, \dots, u_m$, that is, $\{v_1, \dots, v_k, u_1, \dots, u_m\}$ is a basis for S_1. Similarly, by adding extra vectors, say, $\omega_1, \dots, \omega_\ell$ to $\{v_1, \dots, v_k\}$, we can form a basis for S_2: $\{v_1, \dots, v_k, \omega_1, \dots, \omega_\ell\}$ is a basis for S_2. We claim that $B := \{v_1, \dots, v_k, u_1, \dots, u_m, \omega_1, \dots, \omega_\ell\}$ forms a basis for $S_1 + S_2$. Suppose $x \in S_1 + S_2$. Then $x = x_1 + x_2$, where $x_1 \in S_1$, $x_2 \in S_2$. But $\{v_1, \dots, v_k, u_1, \dots, u_m\}$, $\{v_1, \dots, v_k, \omega_1, \dots, \omega_\ell\}$ form bases for S_1, S_2, respectively. Hence there exist scalars $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m, \alpha_1', \dots, \alpha_k', \gamma_1, \dots, \gamma_\ell$ such that

$$x_1 = \alpha_1 v_1 + \dots + \alpha_k v_k + \beta_1 u_1 + \dots + \beta_m u_m,$$

$$x_2 = \alpha_1' v_1 + \dots + \alpha_k' v_k + \gamma_1 w_1 + \dots + \gamma_\ell w_\ell .$$

Thus $x = x_1 + x_2 = (\alpha_1 + \alpha_1')v_1 + \dots + (\alpha_k + \alpha_k')v_k + \beta_1 u_1 + \dots + \beta_m u_m + \gamma_1 w_1 + \dots + \gamma_\ell w_\ell$

$$\in \operatorname{span}\{v_1, \dots, v_k, u_1, \dots, u_m, w_1, \dots, w_\ell\}$$

So $S_1 + S_2 \subset \operatorname{span} B$.

Now if $x \in \operatorname{span} B$, then there exist scalars $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_\ell$ such that $x = \underbrace{\alpha_1 v_1 + \dots + \alpha_k v_k + \beta_1 u_1 + \dots + \beta_m u_m}_{\in S_1} + \underbrace{\gamma_1 w_1 + \dots + \gamma_\ell w_\ell}_{\in S_2}$

Thus $x \in S_1 + S_2$. Consequently $S_1 + S_2 = \operatorname{span} B$. Now we show that B is linearly independent. Suppose

$$\alpha_1 v_1 + \dots + \alpha_k v_k + \beta_1 u_1 + \dots + \beta_m u_m + \gamma_1 w_1 + \dots + \gamma_\ell w_\ell = 0 .$$

Then $\underbrace{\alpha_1 v_1 + \dots + \alpha_k v_k + \beta_1 u_1 + \dots + \beta_m u_m}_{\in S_1} = \underbrace{-(\gamma_1 w_1 + \dots + \gamma_\ell w_\ell)}_{\in S_2}$, and so

there exist scalars $\alpha_1', \dots, \alpha_k'$ such that

$$-(\gamma_1 w_1 + \dots + \gamma_\ell w_\ell) = \alpha_1' v_1 + \dots + \alpha_k' v_k .$$

But $\{v_1, \dots, v_k, w_1, \dots, w_\ell\}$ is independent. Hence $\gamma_1 = \dots = \gamma_\ell = 0$.

Then $\alpha_1 v_1 + \dots + \alpha_k v_k + \beta_1 u_1 + \dots + \beta_m u_m = 0$. By the independence of $\{v_1, \dots, v_k, u_1, \dots, u_m\}$, also $\alpha_1 = \dots = \alpha_k = 0$ and $\beta_1 = \dots = \beta_m = 0$.

Hence
$$\begin{aligned}\dim(S_1 + S_2) &= k + m + \ell \\ &= (k+m) + (k+\ell) - k \\ &= \dim S_1 + \dim S_2 - \dim(S_1 \cap S_2) .\end{aligned}$$