

Exercise 26.5

$$x^T H x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 - x_2 \\ -x_1 + 4x_2 \end{bmatrix}$$

$$= x_1^2 - x_1 x_2 - x_1 x_2 + 4x_2^2$$

$$= x_1^2 - 2x_1 x_2 + x_2^2 + 3x_2^2$$

$$= (x_1 - x_2)^2 + 3x_2^2$$

$$\geq 0$$

and if  $x^T H x = 0$ , then  $(x_1 - x_2)^2 + 3x_2^2 = 0$ , and so

$x_1 = x_2$  and  $x_2 = 0$  i.e.,  $x = 0$ .

Hence  $H$  is positive definite.

Exercise 26.11

(1) If  $1 \leq k \leq n$ , then take  $x \in \mathbb{R}^n$  of the form

$$x = \begin{bmatrix} \xi \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ where } \xi \in \mathbb{R}^k. \text{ Denote the } k \times k \text{ principal submatrix of } H \text{ by } H_k.$$

$$\text{Then } 0 \leq x^T H x = \begin{bmatrix} \xi^T & 0^T \end{bmatrix} \begin{bmatrix} H_k & * \\ * & * \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix} = \begin{bmatrix} \xi^T & 0^T \end{bmatrix} \begin{bmatrix} H_k \xi \\ * \end{bmatrix} = \xi^T H_k \xi$$

Hence  $\xi^T H_k \xi \geq 0 \quad \forall \xi \in \mathbb{R}^k$ . Hence  $H_k$  is positive semidefinite. Also if  $\xi^T H_k \xi = 0$ , then  $x^T H x = 0$ , where  $x := \begin{bmatrix} \xi \\ 0 \end{bmatrix}$ . By the positive definiteness of  $H$ ,  $x = 0$ . Hence  $\xi = 0$ . So  $H_k$  is positive definite. By Property 26.11, it follows that all eigenvalues of  $H_k$  are positive, and so  $\det H_k$  is positive too.

(Here we use the fact that for a diagonalizable matrix  $A$ ,  $\det A = \text{product of its eigenvalues}$ .)

Indeed one way to see this is as follows: if  $A = PDP^{-1}$ , then  $\det A = \det (PDP^{-1}) = (\det P)(\det D)(\det P^{-1}) = (\det D)(\det (PP^{-1})) = \det D.$ )

(2) Let  $S_1 := W$  and  $S_2 := \text{span} \{v_{m+1}, \dots, v_n\}$ .

Then by the result of Exercise 23.9, we have

$$\begin{aligned} \dim(S_1 \cap S_2) &= -\dim(S_1 + S_2) + \dim S_1 + \dim S_2 \\ &\geq -n + k + n - m \\ &= k - m > 0. \end{aligned}$$

Hence  $S_1$  has a nontrivial intersection with  $S_2$ , that is, there is a nonzero vector in  $W$  which is a linear combination of  $v_{m+1}, \dots, v_n$ .

(3) Suppose that  $\omega^T H \omega > 0$  for all nonzero vectors  $\omega$  in a  $k$ -dimensional subspace  $W$  of  $\mathbb{R}^n$ . By the spectral theorem,  $H$  has an orthonormal basis of eigenvectors  $v_1, \dots, v_n$ . Suppose that the first  $m$  of these eigenvectors are the ones corresponding to positive eigenvalues, while the others correspond to nonpositive eigenvalues  $\lambda_{m+1}, \dots, \lambda_n$ . Suppose that  $m < k$ . Then by the previous part of this exercise, there is a nonzero  $\omega$  which is a linear combination of  $v_{m+1}, \dots, v_n$ , say

$$\omega = \alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n.$$

$$\begin{aligned} \text{Then } \omega^T H \omega &= (\alpha_{m+1} v_{m+1}^T + \dots + \alpha_n v_n^T) (\alpha_{m+1} \lambda_{m+1} v_{m+1} + \dots + \alpha_n \lambda_n v_n) \\ &= \alpha_{m+1}^2 \lambda_{m+1} + \dots + \alpha_n^2 \lambda_n \\ &\leq 0, \end{aligned}$$

a contradiction. Hence  $m \geq k$ .

(4). If  $H$  is  $1 \times 1$ , then this is obvious: since  $x^T H x = x^2 (\det H) > 0$  for all nonzero  $x \in \mathbb{R}^1$ .

Suppose the result is true for  $n \times n$  matrices.

Now let  $H \in \mathbb{R}^{(n+1) \times (n+1)}$  be such that all principal minors of  $H$  are positive definite. In particular, all  $n$  principal minors of  $H_n$  are positive definite, and so  $H_n$  is positive definite by the induction hypothesis. (Here  $H_n$  denotes the  $n$ th principal submatrix of  $H$ .)

Let  $W = \text{span} \{e_1, \dots, e_n\}$ . Then if  $\omega \in W$ , we have  $\omega = \begin{bmatrix} \xi \\ 0 \end{bmatrix}$ , where  $\xi \in \mathbb{R}^n$ . Hence  $\omega^T H \omega = \xi^T H_n \xi > 0$  if  $\xi \neq 0$ . Hence  $\omega^T H \omega > 0$  for all nonzero vectors  $\omega$  in  $W$ .

By the previous part of this exercise, it follows that at least  $n$  eigenvalues of  $H$  must be positive. But the product of all  $n+1$  eigenvalues of  $H$  is equal to  $\det H > 0$ . Hence all eigenvalues of  $H$  are positive, and it follows that  $H$  is positive definite.

### Exercise 26.13

$$\det A_1 = \det [4] = 4 > 0,$$

$$\det A_2 = \det \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} = 12 - 4 = 8 > 0,$$

$$\det A_3 = \det A = \det \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix} = 4(6-1) - 2(4+1) + 1(-2-3)$$

$$= 20 - 10 - 5 = 5 > 0.$$

Hence  $A$  is positive definite.

$$\det B_1 = \det [3] = 3 > 0,$$

$$\det B_2 = \det \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} = 12 - 1 = 11 > 0,$$

$$\text{but } \det B_3 = \det B = \det \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix} = 3(4-4) + 1(-1+4) + 2(2-8)$$
$$= 3 + (-12) = -9 < 0.$$

So  $B$  is not positive definite.

Clearly  $-A$  is not positive definite, since for nonzero  $x$ ,  $x^T(-A)x = -x^T A x < 0$ .

Since  $A$  is positive definite, its eigenvalues are all positive. But the eigenvalues of  $A^3$ , being the cubes of these, are also positive. So  $A^3$  is positive definite.

Finally the eigenvalues of  $A^{-1}$  are reciprocals of the eigenvalues of  $A$ , and so  $A^{-1}$  is positive definite as well. (Or directly:  $\forall y, \exists! x$  s.t.  $Ax = y$ .)

So  $y^T A^{-1} y = x^T A A^{-1} A x = x^T A x > 0$  for all nonzero  $y$ .)

Exercise 26.14

(1) TRUE. (If  $A = P^T D P$ , then  $A^5 = P^T D^5 P$ .  $D$  has positive entries, and so does  $D^5$ .)

(2) FALSE (Take for example,  $A = -I$ . Then  $A^8 = I$ .)

(3) TRUE. (If  $A = P^T D P$ , then  $D$  has negative entries.  
Then  $A^{12} = P^T D^{12} P$ , and  $D^{12}$  has positive entries.)

(4) TRUE. (Let  $x \neq 0$ . Then  $x^T A x > 0$ ,  $x^T B x \leq 0$ .  
Hence  $x^T (A - B) x = x^T A x - x^T B x > 0$ .)

Exercise 26.23

$$H = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

Let  $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . Then  $E_1 H E_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$

Let  $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ . Then  $E_2 E_1 H E_1^T E_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ .

So  $H$  is neither.

Let  $E'_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . Then  $E'_1 H E_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix}$ .

Let  $E'_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$ . Then  $E'_2 E'_1 H E_1^T E_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ .

So  $H'$  is positive semi definite.