Exercise 8.2

Let $x, y \in \mathbb{R}$ and $t \in (0,1)$. Then we have

$$
\begin{aligned}
f((1-t)x + ty) &= ((1-t)x + ty)^2 \\
&= (1-t)^2 x^2 + 2(1-t)t\,xy + t^2 y^2 \\
&= (1-t)x^2 + ty^2 + \left((1-t)^2 - (1-t)\right)x^2 + 2(1-t)txy + (t^2-t)y^2 \\
&= (1-t)x^2 + ty^2 + (1-t)(\cancel{1} - t - \cancel{1})x^2 + 2(1-t)txy + t(t-1)y^2 \\
&= (1-t)x^2 + ty^2 - t(1-t)x^2 + 2t(1-t)xy - t(1-t)y^2 \\
&= (1-t)x^2 + ty^2 - t(1-t)(x^2 - 2xy + y^2) \\
&= (1-t)x^2 + ty^2 - \underbrace{t(1-t)\ (x-y)^2}_{\geq 0} \\
&\leq (1-t)x^2 + ty^2 \\
&= (1-t)f(x) + t f(y).
\end{aligned}
$$

So f is convex.

Exercise 8.3.

Since $f''(x) \geqslant 0$ for all x, it follows that if $a<b$, then $f'(a) \leq f'(b)$. Indeed $f'(b)-f'(a) = \int_a^b f''(x)\,dx \geqslant 0$.

Let $x, y \in \mathbb{R}$ be such that $x<y$, and let $t \in (0,1)$. Define $c = (1-t)x + ty \in (x,y)$. By the Mean-Value Theorem, $\frac{f(c)-f(x)}{c-x} = f'(c_1)$ for some $c_1 \in (x,c)$. Similarly $\frac{f(y)-f(c)}{y-c} = f'(c_2)$ for some $c_2 \in (c,y)$. As $c_2 > c_1$, we have $f'(c_2) \geqslant f'(c_1)$, and so

$$\frac{f(y)-f(c)}{y-c} = \frac{f(y)-f(c)}{y-(1-t)x-ty} \underset{(*)}{\geqslant} \frac{f(c)-f(x)}{c-x} = \frac{f(c)-f(x)}{(1-t)x+ty-x}$$

i.e.,
$$\frac{f(y)-f(c)}{(1-t)(y-x)} \geqslant \frac{f(c)-f(x)}{t(y-x)}.$$

Thus $t f(y) - t f(c) \geqslant (1-t) f(c) - (1-t) f(x)$, that is

$$f(c) = f((1-t)x+ty) \leq (1-t)f(x) + t f(y).$$

So f is convex.

If $f''(x) > 0$ for all x, then note that for $a<b$,

$$f'(b)-f'(a) = \int_a^b f''(x)\,dx > 0,$$

and so $f'(b) > f'(a)$. So in the proof above, we may replace "$\geqslant$" by '$>$' in $(*)$ and all subsequent occurrences, showing the strict convexity of f.

Exercise 8.4

(1) $f'(x) = 1$ and so $f''(x) = 0 \geq 0$ for all x. Hence f is convex.

(2) $f'(x) = 2x$ and so $f''(x) = 2 > 0$ for all x. Hence f is convex.

(3) $f'(x) = e^x$ and so $f''(x) = e^x > 0$ for all x. Hence f is convex.

(4) $f'(x) = -e^{-x}$ and so $f''(x) = e^{-x} > 0$ for all x. Hence f is convex.

(5) If $x, y \in \mathbb{R}$ and $t \in (0,1)$, then

$$\begin{aligned} f((1-t)x + ty) = |(1-t)x + ty| &\leq |(1-t)x| + |ty| \\ &= |1-t||x| + |t||y| \\ &= (1-t)|x| + t|y| \quad (\because t \in (0,1)) \\ &= (1-t)f(x) + t f(y). \end{aligned}$$

Hence f is convex.

In (2), (3), (4), $f''(x) > 0$ for all x and so f is strictly convex. In (1) and (5), the functions are not strictly convex; for example take $x = 1$, $y = 2$ and $t = \frac{1}{2}$. Then

$$|(1-t)x + ty| = (1-t)x + ty = (1-t)|x| + t|y|.$$

Exercise 8.5

Let $x, y \in K$ and $t \in (0,1)$. Then $f(x) \leq r$ and $f(y) \leq r$. Thus

$$f((1-t)x + ty) \leq (1-t) f(x) + t f(y) \leq (1-t) r + t r$$
$$= r.$$

So $(1-t)x + ty \in K$. Hence K is convex.

Exercise 8.6.

Let (x, α), $(y, \beta) \in U(f)$.

Then $\alpha > f(x)$ and $\beta > f(y)$.

If $t \in (0,1)$, then

$$(1-t)\alpha + t\beta > (1-t)f(x) + tf(y) \geq f((1-t)x + ty)$$

and so $(1-t)(x, \alpha) + t(y, \beta) = ((1-t)x + ty, (1-t)\alpha + t\beta) \in U(f)$.

Hence $U(f)$ is convex.

Exercise 8.7

We prove this using induction on n. The inequality is true when $n=1$, since $f\left(\frac{x_1}{1}\right) = \frac{f(x_1)}{1}$, $x_1 \in C$.

Suppose the result is true for some n. Let $x_1, \dots, x_n, x_{n+1} \in C$. Then by the result in Exercise 4.13, $\frac{x_1+\dots+x_n}{n}$, $\frac{x_1+\dots+x_n+x_{n+1}}{n+1}$ belong to C. We have

$$
\begin{aligned}
f\left(\frac{x_1+\dots+x_n+x_{n+1}}{n+1}\right) &= f\left(\frac{x_1+\dots+x_n}{n+1} + \frac{x_{n+1}}{n+1}\right)\\
&= f\left(\frac{n}{n+1}\cdot\frac{x_1+\dots+x_n}{n} + \frac{1}{n+1}x_{n+1}\right)\\
&= f\left(\left(1-\frac{1}{n+1}\right)\frac{x_1+\dots+x_n}{n} + \underbrace{\frac{1}{n+1}}_{=:t}x_{n+1}\right)\\
&\leq \left(1-\frac{1}{n+1}\right) f\left(\frac{x_1+\dots+x_n}{n}\right) + \frac{1}{n+1}f(x_{n+1}) \quad \text{(convexity of } f\text{)}\\
&\leq \frac{n}{n+1}\,\frac{f(x_1)+\dots+f(x_n)}{n} + \frac{1}{n+1}f(x_{n+1}) \quad \text{(induction hypothesis)}\\
&= \frac{f(x_1)+\dots+f(x_n)}{n+1} + \frac{f(x_{n+1})}{n+1}\\
&= \frac{f(x_1)+\dots+f(x_n)+f(x_{n+1})}{n+1}.
\end{aligned}
$$

So the result holds for all n.

Exercise 8·8

(i) TRUE.

For if $x, y \in C_1$ and $t \in (0,1)$, then $x, y \in C_2$ and $(1-t)x + ty \in C_1$

$$f((1-t)x + ty) = F((1-t)x + ty) \leq (1-t)F(x) + t\,F(y)$$
$$= (1-t)f(x) + t f(y).$$

(ii) FALSE.

For example, take $C_1 := (-1, 1) \subset C_2 := \mathbb{R}$, $f(x) = x$,

$$F(x) = \begin{cases} x & \text{if } |x| \leq 1, \\ 1 & \text{if } x > 1, \\ -1 & \text{if } x < -1. \end{cases}$$

Then F is not convex

(For example, take $x = 0$, $y = 2$, $t = 1/2$.

Then $F((1-t)x + ty) = F(1) = 1 > \frac{1}{2} = (1-t)\underbrace{F(x)}_{0} + t\,\underbrace{F(y)}_{1}$.)

Exercise 8.9.

(1) Let $x, y \in C$, $t \in (0,1)$ and $\beta \in I$. Then

$$f_\beta((1-t)x + ty) \leq (1-t) f_\beta(x) + t f_\beta(y)$$
$$\leq (1-t) \sup_{\alpha \in I} f_\alpha(x) + t \sup_{\alpha \in I} f_\alpha(y)$$
$$= (1-t) f(x) + t f(y).$$

Thus $\sup_{\beta \in I} f((1-t)x + ty) \leq (1-t) f(x) + t f(y)$

i.e., $\quad f((1-t)x + ty) \leq (1-t) f(x) + t f(y)$.

Hence f is convex.

By the result from Exercise 8.5 (with $r=0$), we have that

$$\{x \in C : \sup_{\alpha \in I} f_\alpha(x) = f(x) \leq 0\} \quad (*)$$

is convex. But if $\sup_{\alpha \in I} f_\alpha(x) \leq 0$, then $f_\alpha(x) \leq 0$ for all $\alpha \in I$. Conversely if $f_\alpha(x) \leq 0$ for all $\alpha \in I$, then $\sup_{\alpha \in I} f_\alpha(x)$. Consequently, for $x \in C$, we have

$$\left[\sup_{\alpha \in I} f_\alpha(x) \leq 0\right] \iff \left[f_\alpha(x) \leq 0 \text{ for all } \alpha \in I\right].$$

So the set $(*)$ is the same as

$$K = \{x \in C : f_\alpha(x) \leq 0 \text{ for all } \alpha \in I\}.$$

(2) Let $x, y \in C$ and $t \in (0,1)$. Then

$$s((1-t)x + ty) = \sum_{i=1}^n \alpha_i f_i((1-t)x + ty)$$
$$\leq \sum_{i=1}^n \alpha_i \big((1-t) f_i(x) + t f_i(y)\big) \quad \text{(since } f_i \text{ is conve[x] and since } \alpha_i \geq 0 \text{ fo[r] each } i)$$
$$= (1-t) \sum_{i=1}^n \alpha_i f_i(x) + t \sum_{i=1}^n \alpha_i f_i(y)$$
$$= (1-t) s(x) + t s(y).$$

So s is convex.