EXAM FOR OPTIMIZATION SF1811/SF1831/SF1841

JUNE 7, 2010

Examiner: Amol Sasane (Phone: 790 7320)
Writing time allowed: 1400-1900
Writing material: Pen or pencil, eraser and a ruler is allowed. Calculators are **not** allowed! A formula sheet is provided.
Instructions: Motivate your answers carefully. Write your name on each page of the solutions you hand in and number the pages. Write the solutions to the different questions 1,2,3,4,5 on separate sheets. A passing grade is guaranteed for 25 points (including bonus points from the voluntary home assignments).

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(1) Consider the following linear programming problem:

$$(P): \begin{cases} \text{minimize} & 7x_1 + 4x_2 + 8x_3 + 6x_4 \\ \text{subject to} & 2x_1 + x_2 + 3x_3 + x_4 = 7, \\ & x_1 + 2x_2 + x_3 + 3x_4 = 7, \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0. \end{cases}$$

(a) Show that the following solution is optimal for (P):

$$x_1 = 0, \quad x_2 = \frac{14}{5}, \quad x_3 = \frac{7}{5}, \quad x_4 = 0.$$

(b) Write down the dual problem (D) to (P).

(c) Sketch the feasible region for (D) and solve the problem (D) graphically. Explain why the optimal solution you find for (D) is expected, based on the calculations done in part (a).

(d) Suppose that in the problem (P), the cost vector changes from

$$c := \begin{bmatrix} 7 & 4 & 8 & 6 \end{bmatrix}^\top$$

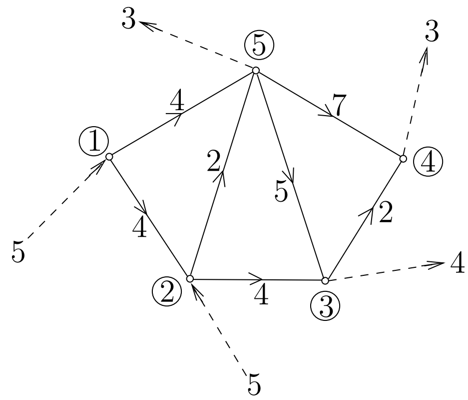
to the new vector

$$\widetilde{c} := \begin{bmatrix} 7+\delta & 4+\delta & 8+\delta & 6+\delta \end{bmatrix}^\top,$$

where δ is a real number. For what values of δ is the solution in part (a) still optimal for the problem?

(12 points)

(2) Consider the minimum cost of flow problem for the network shown below.



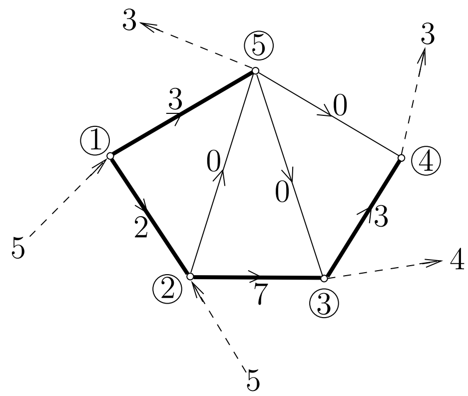
We have numbered the 5 nodes. The nodes 1 and 2 are source nodes with a supply of 5 units each, while the nodes 3, 4 and 5 are sink nodes with demands of 4, 3 and 3 units, respectively. Beside each directed edge we have indicated the cost c_{ij} per unit flow.

(a) Write the incidence matrix A for the network, with the following order of the edges:

$$(1,2),\ (1,5),\ (2,3),\ (2,5),\ (3,4),\ (5,3),\ (5,4).$$

(Above, the notation (i,j) means the directed edge from node i to node j.) Let x_{ij} denote the flow from node i to node j. Specify the constraints on variables x_{ij} of the linear programming problem associated with this network flow problem.

(b) Show that the solution in the figure below is optimal. We have indicated the flow x_{ij} beside each edge (i,j).



(c) Suppose that c_{53} changes from 5 to 3. Verify that the solution in part (a) is no longer optimal. Hence determine an optimal solution to the new problem (with $c_{53} = 3$). Start from the solution given in part (a).

(12 points)

(3) (a) Suppose that $C \subset \mathbb{R}^n$ is a convex set. What does it mean to say that $f : C \to \mathbb{R}$ is a convex function?

(b) For which real values of a is the following function convex on $\mathbb{R}^3$?

$$f(x_1, x_2, x_3) = x_1^2 + 5x_2^2 + ax_3^2 + 2x_1x_2 + 4x_2x_3 + x_2^4, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

(c) Find all global minimizers for the function g on $\mathbb{R}^2$ given by

$$g(x_1, x_2) = x_1^4 - 12x_1x_2 + x_2^4, \quad (x_1, x_2) \in \mathbb{R}^2.$$

(2+5+5=12 points)

(4) Consider the following optimization problem:

$$(NP) : \begin{cases} \text{minimize} & x_1 \\ \text{subject to} & x_2 - x_1^3 \leq 0, \\ & -x_2 - x_1^3 \leq 0. \end{cases}$$

(a) Depict the feasible region graphically. In the same figure, show the level curves of the objective function using dotted lines. Hence conclude that the origin $(0, 0) \in \mathbb{R}^2$ is a global minimizer.

(b) Write the KKT optimality conditions corresponding to the problem (NP), and show that they are not satisfied at the point $(0, 0)$. Explain this by considering the regularity of the point $(0, 0)$.

(7 points)

(5) Consider the quadratic optimization problem

$$(QP) : \begin{cases} \text{minimize} & x_1^2 - x_1x_2 + x_2^2 \\ \text{subject to} & x_1 - x_2 = 1. \end{cases}$$

(a) Find an optimal solution to (QP) using the nullspace method.

(b) Suppose that the equality in the constraint is replaced by $\leq$, so that the new constraint is $x_1 - x_2 \leq 1$. Use the Lagrange relaxation method to solve the new problem:

$$(QP') : \begin{cases} \text{minimize} & x_1^2 - x_1x_2 + x_2^2 \\ \text{subject to} & x_1 - x_2 \leq 1. \end{cases}$$

(7 points)

End of the exam.