Exam January 10, 2018 in SF1811 Optimization.

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Allowed utensils: Pen, paper, eraser and ruler. A formula-sheet is handed out. No calculator! No books or notes.

Language: Your solutions should be written in English or in Swedish.

Solution methods: All conclusions should be properly motivated. Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.

Note: Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Consider the following linear programming problem:

$$(LP): \begin{cases} \min & -x_2 + 2x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 + x_5 = 4 \\ & x_1 - x_2 + x_3 + x_4 = 6 \\ & x_i \geq 0, \text{ for } i = 1,2,3,4,5. \end{cases}$$

   (i) Use the Simplex method to compute an optimal solution of (LP). Start with x_4 and x_5 as basic variables. .... (7p)

   (ii) Explicitly write down the dual problem to (LP). Determine an optimal solution to the dual problem and verify that this is optimal, e.g., by using the optimal solution to (LP). .... (3p)

2. Let $f(x) = \frac{1}{2}x_1^2x_2^2 + 4x_1^2 + x_2^2 - 5x_1x_2 - 2x_2$ where $x = (x_1, x_2)^T \in \mathbf{R}^2$, and consider the problem of minimizing $f(x)$ without any constraints.

   (i) Use $x^{(1)} = (2,2)^T$ as starting point and calculate the next iteration point $x^{(2)}$ using Newtons method...................... (5p)

   (ii) Is the function $f(x)$ convex on the set $\mathbf{R}^2$? Motivate! ..... (2p)

   (iii) Assume that the function should be minimized subject to the constraints $x_1 = x_2$. Find the optimal solution to this constrained problem (using a method of your choice). Show that the solution is a global optimum. *(Hint: Note that a twice differentiable* *function f is convex on a convex set $C \subset \mathbf{R}^n$ if and only if $(x-y)^T F(x)(x-y) \geq 0$ for all $x, y \in C$, where $F(x)$ denotes the Hessian of f at x.)* .... (3p)

3. a) Consider the matrix

$$H = \begin{pmatrix} 16 & 8 & 4 \\ 8 & 4 & 2 \\ 4 & 2 & 1 \end{pmatrix}$$

(i) Determine if H is positive semidefinite or not, e.g., using LDL^T-factorization.

(ii) Verify that $c = (0, -1, 2)^T$ belongs to the kernel of H.

(iii) Consider the unconstrained quadratic optimization problem

$$(P): \min_{x \in \mathbf{R}^3} \frac{1}{2} x^T H x + c^T x.$$

Determine the optimal solution of (P) (if such solution exist). Justify your answer.

.... (5p)

b) Consider the portfolio optimization problem, where we seek the portfolio with minimal variance among the portfolios with given expected profit. Let $x \in \mathbf{R}^n$ where x_i is the number of shares in asset i. Let $r \in \mathbf{R}^n$ where r_i is the expected profit of asset i and let $[c_{ij}]_{i,j=1}^n = C \in \mathbf{R}^{n \times n}$ be a positive definite matrix where c_{ij} is the covariance of the profit of asset i and asset j (variance if $i = j$). Solve the following optimization problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & x^T C x \\ \text{subject to } & r^T x = 1. \end{aligned}$$

Determine the optimal allocation x and the optimal value (minimal variance) explicitly in r and C. *(Hint: Use the Lagrange method.)* .... (5p)

4. (a) Here we will consider an flow problem where we seek to transport electricity in a network with minimal losses. Consider the network in Figure 1 with five nodes and six edges, and where one unit of current enters the network at node A (A is a source) and one unit of current exits the system at node E (E is a sink). The loss in an edge is the square of the current in that edge.

(i) Formulate the problem of minimizing the sum of all losses in the network (i.e., sum of losses in all edges) as a QP problem with constraints. The flow in an edge is allowed to go in both directions. .... (5p)

(ii) Solve the optimization problem (e.g., using the nullspace method). . . . . (5p)

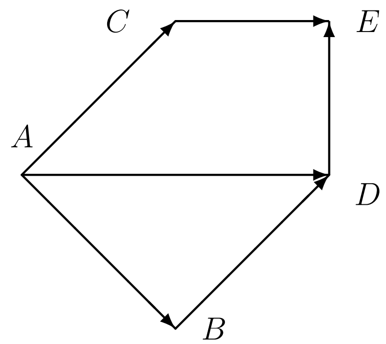

Figure 1: Flow network.

5. Consider a setup where several communication channels are available and where we seek to maximize the transmitted information content subject to a constraint on the total transmission power. The capacity of channel i is (by Shannon-Hartley theorem)

$$B_i \log(1 + x_i/N_i)$$

where $B_i > 0$ relates to the bandwidth, x_i is the transmission power, and $N_i > 0$ is the noise power. We want to determine the how to distribute the transmission power over the channels so that the total capacity is maximized when the total transmission power is bounded by P, i.e., to solve the optimization problem

$$(P): \begin{cases} \max\limits_{x_i, i=1,\ldots,n} & \sum_{i=1}^{n} B_i \log(1 + x_i/N_i) \\ \text{subject to} & \sum_{i=1}^{n} x_i \leq P \\ & x_i \geq 0, \ \text{for } i = 1, \ldots, n. \end{cases}$$

(i) Relax the power constraint (Lagrange relaxation) and compute the optimal power configuration as a function of the Lagrange multiplier (i.e., compute $\hat{x}_i(y)$). . . . . (6p)

(ii) Consider the special case: $n = 3$, $B_1 = B_2 = B_3 = 1$, $N_1 = 1, N_2 = 3, N_3 = 5$, and $P = 4$. Find the Lagrange multiplier so that the global optimality conditions are satisfied. . . . . (4p)

Good luck!

1 Solutions

1. a) Let $\beta = (4,5)$ and $\nu = (1,2,3)$. This gives

$$A_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c_\beta = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad A_\nu = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad c_\nu = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

Hence we have

$$A_\beta \bar{b} = b = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \Rightarrow \bar{b} = \begin{pmatrix} 6 \\ 4 \end{pmatrix},$$

which is a basic feasible solution. Next, we have

$$A_\beta^T y = c_\beta \Rightarrow y = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Therefore

$$r_\nu = c_\nu - A_\nu^T y = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

Since the basic feasible solution $\bar{b}$ is strictly positive and the second component of the reduced cost is negative the solution is not optimal. Therefore we introduce the variable $\nu_2 = 2$ as new active variable. Noting that $\bar{a}_2 = (-1,2)^T$ we get $t_{\max} = 2$ which gives $x_{\beta_2} = 0$ and hence the variable x_5 can be removed from the basic tuple.

Next we will do a second simplex iteration. This time for the basic tuple $\beta = (2,4)$:

$$A_\beta = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}, \quad c_\beta = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad A_\nu = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad c_\nu = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

Hence we have

$$A_\beta \bar{b} = b = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \Rightarrow \bar{b} = \begin{pmatrix} 2 \\ 8 \end{pmatrix},$$

which is a basic feasible solution. Next, we have

$$A_\beta^T y = c_\beta \Rightarrow y = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Therefore

$$r_\nu = c_\nu - A_\nu^T y = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

Since the the reduced cost is positive the solution $x = (0, 2, 0, 8, 0)^T$ is the optimal solution.

b) The dual problem is given by

$$(D): \begin{cases} \max & 4y_1 + 6y_2 \\ \text{subject to} & y_1 + y_2 \le 0 \\ & 2y_1 - y_2 \le -1 \\ & y_2 \le 2 \\ & y_1 \le -1 \\ & y_1 \le 0. \end{cases}$$

By complementarity $x^T(c - A^T y) = 0$, hence the second and forth component of $c - A^T y$ must be zero, i.e., $2y_1 - y_2 = -1$ and $y_1 = -1$. This gives the optimal solution $y_1 = -1$, $y_2 = -1$. Also note that this is a feasible solution to the dual problem and that the corresponding values of the objective function is -10 for both the primal and dual problem.

2. a) Note that

$$\nabla f(x) = \begin{pmatrix} x_1 x_2^2 + 8x_1 - 5x_2 \\ x_1^2 x_2 + 2x_2 - 5x_1 - 2 \end{pmatrix}, \quad H(x) = \begin{pmatrix} x_2^2 + 8 & 2x_1 x_2 - 5 \\ 2x_1 x_2 - 5 & x_1^2 + 2. \end{pmatrix}$$

In the point $x^{(2)} = (2, 2)^T$ we have

$$\nabla f(x^{(2)}) = \begin{pmatrix} 14 \\ 0 \end{pmatrix}, \quad H(x^{(2)}) = \begin{pmatrix} 12 & 3 \\ 3 & 6 \end{pmatrix},$$
$$d = -H(x^{(2)})^{-1} \nabla f(x^{(2)}) = \begin{pmatrix} -4/3 \\ 2/3. \end{pmatrix}$$

With $t = 1$ the new solution is $x^{(1)} + (-4, 2)^T/3 = (2, 8)^T/3$. Finally note that

$$f(x^{(1)}) = 4 > f\left(x^{(1)} + (-4, 2)^T/3\right) = -304/81 \approx -3.75$$

hence the new point has a smaller objective value than the previous solution, hence we let $x^{(2)} = (2, 8)^T/3$.

b) Note, e.g., that

$$H((0, 0)^T) = \begin{pmatrix} 8 & -5 \\ -5 & 2 \end{pmatrix}$$

is not positive semidefinite, hence the function is not convex on $\mathbf{R}^2$.

c) On the set $x_1 = x_2$, the objective function is

$$\frac{1}{2}x_1^4 - 2x_1.$$

This is convex, and hence any stationary point is an optimal point. Take the derivative with respect to x_1:

$$\frac{\partial}{\partial x_1}\left(\frac{1}{2}x_1^4 - 2x_1\right) = 2x_1^3 - 2$$

and note that this is zero only if $x_1 = 1$ (x_1^3 is increasing). Hence the unique optimal point is $x_1 = x_2 = 1$.
Note that since the set $\{x : x_1 = x_2\}$ have empty interior, the function can be convex even though the Hessian is not positive definite. Convexity can also be verified from $(x-y)^T H(x)(x-y) \geq 0$ for all $x, y \in \{x : x_1 = x_2\}$. In our case this holds since

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1^2 + 8 & 2x_1^2 - 5 \\ 2x_1^2 - 5 & x_1^2 + 2. \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 6x_1^2 \geq 0$$

for any x_1.

3. a) i) Note that

$$H = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}$$

which is positive semidefinite. ii) Note that

$$Hc = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

hence c belong to the kernel of H. iii) Since $c \in \mathcal{N}(H)$ we have that $c \perp \mathcal{R}(H)$ and since $c \neq 0$ it follows that $c \notin \mathcal{R}(H)$. Therefore there is no solution to $Hx + c = 0$ and no solution to the minimization problem. Note that $-c$ is a descent direction and the objective value goes to $-\infty$ for $x = -tc$ as $t \to \infty$.

b) Using the Lagrange method the solution x is optimal if there is an u such that

$$\begin{pmatrix} 2C & -r \\ r^T & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note that C is positive definite this is equivalent to

$$x = \frac{1}{2}C^{-1}ru$$
$$r^T x = 1.$$

These equations together gives $u = 2(r^T(C)^{-1}r)^{-1}$, which yields the optimal solution $x = (r^T C^{-1} r)^{-1} C^{-1} r$ and the optimal value

$$x^T C x = (r^T C^{-1} r)^{-2} r^T C^{-1} C C^{-1} r = (r^T C^{-1} r)^{-1}.$$

4. a) Let x_{ij} be the current in the (directed) edge (i, j) where

$$(i, j) \in E = \{(1,2), (1,3), (1,4), (2,4), (3,5), (4,5)\}.$$

That is, if $x_{ij} > 0$ there is a flow of x_{ij} from node i to node j and if $x_{ij} < 0$ there is a flow of $|x_{ij}|$ from node j to node i. The optimization problem is then given by

$$\begin{cases} \text{min} & x^T x \\ \text{subject to} & Ax = b \end{cases}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} x_{12} \\ x_{13} \\ x_{14} \\ x_{24} \\ x_{35} \\ x_{45} \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

b) Note that

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$$

is a basis for the nullspace of A. This can be seen by noting that the first column of A corresponds to flow in the cycle $(A \to B \to D \to A)$ and the second column the cycle $(A \to C \to E \to D \to A)$. Also note that the flow of one unit current from along the path $(A \to C \to E)$ is represented by

$$x_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}^T,$$

which is therefore a feasible solution. Therefore any solution can be written as $x = x_0 + Zv$, where $v = (v_1, v_2)^T \in \mathbf{R}^2$, any optimal v satisfies

$$(Z^T Z)v = -Z^T x_0.$$

Since $Z^TZ = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$ is positive definite, the optimal solution is given by

$$v = -(Z^TZ)^{-1}Z^Tx_0 = \begin{pmatrix} 2 \\ -6 \end{pmatrix}/11, \quad x = x_0 + Zv = \begin{pmatrix} 2 \\ 5 \\ 4 \\ 2 \\ 5 \\ 6 \end{pmatrix}/11.$$

5. a) We first change sign of the objective, and consider the corresponding minimization problem. For a given $y \geq 0$, the relaxed problem is

$$(PR_y): \begin{cases} \min\limits_{x_i, i=1,\ldots,n} & -\sum_{i=1}^n B_i \log(1 + x_i/N_i) + y(\sum_{i=1}^n x_i - P) \\ \text{subject to} & x_i \geq 0, \text{ for } i = 1, \ldots, n. \end{cases}$$

Note that the objective function can be separated into terms so that each term only depends on one variable, and hence the optimal solution can be computed for each variable separately. That is, the part of the objective function that depends on the variable $x_i \geq 0$ is

$$-B_i \log(1 + x_i/N_i) + yx_i.$$

This is a convex function with derivative $-B_i/(N_i + x_i) + y$. If the derivative at $x_i = 0$ is non-negative, i.e., if $B_i/N_i \leq y$, then the minimizing argument is $x_i = 0$. If the derivative at $x_i = 0$ is negative, then the minimizing argument x_i is characterized by $-B_i/(N_i + x_i) + y = 0$, that is $x_i = B_i/y - N_i$. To summarize, the optimal values $\hat{x}_i(y)$ for $i = 1, \ldots, n$, to the problem (PR_y) are given by

$$\hat{x}_i(y) = \begin{cases} 0 & \text{if } B_i/N_i \leq y \\ B_i/y - N_i & \text{if } B_i/N_i > y. \end{cases}$$

b) For the parameters in our problem we get

$$\hat{x}_1(y) = \begin{cases} 0 & \text{if } 1 \leq y \\ 1/y - 1 & \text{if } 1 > y, \end{cases}$$

$$\hat{x}_2(y) = \begin{cases} 0 & \text{if } 1/3 \leq y \\ 1/y - 3 & \text{if } 1/3 > y, \end{cases}$$

$$\hat{x}_3(y) = \begin{cases} 0 & \text{if } 1/5 \leq y \\ 1/y - 5 & \text{if } 1/5 > y. \end{cases}$$

In order to satisfy GOC, we need $y(\sum_{i=1}^{3} \hat{x}_i(y) - 4) = 0$. Note that $y = 0$ is not feasible to the primal problem since $\hat{x}_i(y) \to \infty$ as $y \to 0$. Therefore we must have $\sum_{i=1}^{3} \hat{x}_i(y) - 4 = 0$. Note that $\hat{x}_i(y)$ are non-increasing functions of y and that $\sum_{i=1}^{3} \hat{x}_i(y) - 4 = 0$ is satisfied for $y = 1/4$. This gives the optimal solution: $\hat{x}_1 = 3$, $\hat{x}_2 = 1$, $\hat{x}_3 = 0$. By definition this satisfies the first and forth criteria of GOC. Primal and dual feasibility also hold, hence we have verified GOC.