KTH Mathematics

Exam in SF1841 Optimization for F.
Tuesday march 11 2008 kl. 08.00–13.00

Instructor: Per Enqvist, tel. 790 62 98

Allowed utensils: Pen, eraser and ruler. **No calculator!** A formula-sheet is handed out.

Solution methods: If not specifically stated in the problem statement, the problems should be solved using systematic methods that do not become futile for large problems. Unless so stated, known theorems can be used without proving them, assuming that they are stated correctly. Motivate your conclusions carefully.

A passing grade is guarranteed for 24 points, including bonus points from the voluntary home assignments. 21-23 points grant the possibility to complement the exam the result within three weeks from the announcement of the results. Contact the instructor if this is the case.

Number the pages of the solutions you hand in. Do not answer more than one problem on any page.

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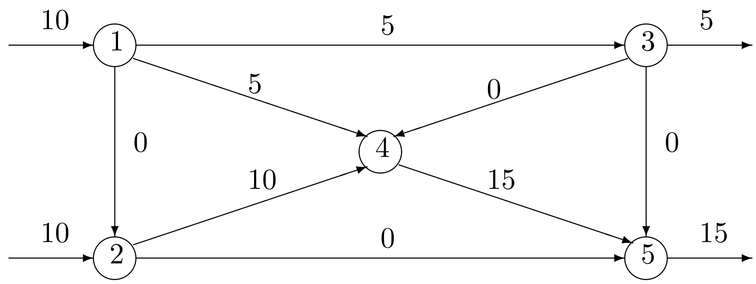
1. (a) Your first assignment is to design the working schedule for the hamburger restaurant "El Burgo". At El Burgo the four employees Adrian, Bobby, Cecile and Davide work and to keep the restaurant going it is necessary to every week spend 40 manhours in the kitchen, 60 manhours at register selling the hamburgers and 40 manhours cleaning tables and the toilets. The employees work at most 40 hours per week each, and they get payed only for the hours (or fraction of an hour) they work. At the last salary negotiations it was decided that Adrian gets 70 SEK/hour, Bobby 80 SEK/hour, Cecile 100 SEK/hour and Davide 110 SEK/hour.

When designing the work shedule the following factors also has to be taken into account. If Adrian has to work with cleaning more hours than Bobby or Cecile, he is going to get very upset and this has to avoided. When Bobby is working at the register 40 SEK/hour usually "Disappears". Cecile is a terrible cook and is not allowed to work in the kitchen, while Davide has a contract that he will work at least 20 hours per week in the kitchen.

You have to formulate the problem to find a working schedule that minimizes the cost for the restaurant while all the criterions mentioned above are satisfied.

The answer should be a linear optimization problem with linear equality and/or inequality constraints, and the variables you use should be defined carefully.
.... (5p)

(b) Consider the following network



Show that the flow suggested in the graph minimizes the cost of the flow if the costs in the links are given by

$$c_{12} = 2, c_{13} = 2, c_{14} = 2, c_{24} = 1, c_{25} = 3, c_{34} = 1, c_{35} = 2, c_{45} = 1.$$

. . . . (3p)

(c) Is the same solution still optimal if the cost c_{25} is changed to $c_{25} = 1$? Motivate your answer.
. . . . (2p)

2. (a) Consider the following linear programing problem

$$(P) \quad \begin{bmatrix} \min\limits_{x} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} 3 & 2 & 1 & 4 \end{bmatrix}^T.$$

Show that the basic solution with x_1 and x_2 being basic variables is feasible. Use this basic solution as a starting point for solving the linear program (P) using the simplex method. . . . . (5p)

(b) Determine the dual linear program (D) to (P). . . . . (2p)

(c) Determine the optimal solution to the dual (D). It is ok to use the result from (a), or a graphic solution of the dual, to come up with a candidate optimal solution as long as optimality is verified using some known theorem. . . . . (3p)

3. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(a) Determine the dimensions of the four fundamental subspaces $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A^T)$, and $\mathcal{R}(A^T)$, and a basis for each of them. .... (4p)

(b) Determine an optimal solution $\hat{\mathbf{x}}$ to the problem to minimize $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T H\mathbf{x}$ subject to the constraint $\mathbf{A}\mathbf{x} = \mathbf{b}$. It could be useful to note that the vector with all elements equal to one is a solution of the linear equation system $\mathbf{A}\mathbf{x} = \mathbf{b}$. .... (4p)

(c) If another linear constraint $ax = \beta$ such that $a\hat{x} = \beta$ is added to the linear program (P), will the point $\hat{x}$ still be optimal ? Motivate your answer. .... (2p)

4. Consider the nonlinear programing problem

$$(P) \quad \left[\begin{array}{ll} \min\limits_{x} & x_1^2 - 4x_1x_2 + 2x_2^2 - 2x_2x_3 - 2x_2 + 3x_3 \\ \text{s.t.} & x_1^2 + x_2^2 + x_3^2 \leq 2 \\ & x1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{array} \right]$$

(a) Show that $x = (0, 0, 0)$ is not a local minimum. .... (2p)

(b) Show that $x = (1, 1, 0)$ satisfies the first order KKT-conditions. .... (3p)

(c) Consider now the unconstrained convex problem

$$(\tilde{P}) \quad \left[\min_{x} \quad x_1^2 - 4x_1x_2 + 6x_2^2 - 8x_2. \right]$$

Start at $x = (0, 0)$ and perform one iteration of Newtons algorithm. Take a full length step to obtain a new point, and show that the value of the objective function decrease. .... (3p)

(d) Motivate why the point determined by Newtons method in (c) is in fact optimal to $(\tilde{P})$. .... (2p)

5. *Maximum Likelihood estimation of a probability distribution*

In statistics it is common to consider outcomes that can be classified into m different categories. We denote the probability to observe an outcome of category i with p_i, and we would like to use a number of independent observations to estimate these probabilities. Assume that we have obtained k_i observations of each category $i = 1, \cdots, m$. Assume that $p_1, p_2, \cdots, p_m$ are known, then

$$L(p) = \prod_{i=1}^{m} p_i^{k_i}$$

is the probability to obtain the ordered observation. The probabilities can then be estimated using the Maximum Likelihood method, which is based on the idea to choose the probabilities in such a way that the likelihood of obtaining the observed outcomes is maximized. It is often easier to consider the logarithm of the likelihood, so we consider the following problem

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{i=1}^{m} k_i \log p_i \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{m} p_i \leq 1, \\ & p_i \geq 0, \quad i = 1, \cdots, m. \end{aligned}$$

The constraints say that the probabilities have to be non-negative and the sum of all the probabilities is less or equal to one. The last condition might seem strange, the natural thing would be to require equality constraint, but this problem is a relaxed version which will generate the same solution.

(a) Determine the dual problem. When you determine the dual, only the constraint on the sum of the probabilities should be relaxed, the positivity constraints should be kept implicit. .... (4p)

(b) Determine a stationary point to the dual problem. .... (2p)

(c) Use the solution in (b) to the dual problem to determine a solution to the primal problem .... (2p)

(d) Motivate global optimality for the primal and dual solutions. .... (2p)

Good luck!