KTH Mathematics

Exam in SF1841 Optimization for F.
Friday may 30 2008 kl. 08.00–13.00

Instructor: Per Enqvist, tel. 790 62 98

Allowed utensils: Pen, eraser and ruler. **No calculator!** A formula-sheet is handed out.

Solution methods: If not specifically stated in the problem statement, the problems should be solved using systematic methods that do not become futile for large problems. Unless so stated, known theorems can be used without proving them, assuming that they are stated correctly. Motivate your conclusions carefully.

A passing grade is guaranteed for 24 points, including bonus points from the voluntary home assignments. 21-23 points grant the possibility to complement the exam the result within three weeks from the announcement of the results. Contact the instructor if this is the case.

Number the pages of the solutions you hand in. Do not answer more than one problem on any page.

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1. (a) Nutty Inc. produce and sell two mixes of nuts: The Luxury mix and the Family mix. Both mixes are made up from hazel-nuts and peanuts. According to the description the luxury mix should contain between 65% and 80% of hazel-nuts (and thus between 20% and 35% of peanuts), while the family mix should contain between 25% and 40% of hazel-nuts (and thus between 60% and 75% of peanuts).

Nutty's purchase price for hazel-nuts is 20 SEK/kilo, and for peanuts it is 10 SEK/kilo. The supply of nuts that is available for Nutty during the considered planning period is limited to 600 kilo of hazel-nuts and 500 kilo of peanuts.

Nutty sells the luxury mix for 36 SEK/kilo and the family mix for 24 SEK/kilo. The demand is not assumed to be a limiting factor, i.e. Nutty is able to sell all of his products at the given prices.

Nutty is now faced with the problem to decide how much of each mix is going to be produced, and how each mix should be composed, during the considered planning period. The objective is to maximize the income from the sales minus the expenses for purchasing the nuts.

Your assignment is to *formulate* the problem of Nutty as an LP-problem.
You should *not* determine the optimal solution.
Motivate carefully your formulation and define your variables properly. . .(6p)

(b) Let f be a function from $I\!R^4$ to $I\!R$ defined by

$$f(\mathbf{x}) = x_1^2 + 2x_1x_2 + 1.5\,x_2^2 + x_2x_3 + x_3^2 + x_3x_4 + x_4^2 + x_4.$$

Determine if $\hat{\mathbf{x}} = (1, -1, 1, -1)^\mathsf{T}$ is a global minimizer, or not, for f. . . . . (4p)

2. (a) Consider the following LP-problem on standard form:

$$\begin{array}{lll} \text{P1:} & \text{minimize} & x_1 + 3x_2 + 2x_3 + x_4 \\ & \text{s.t.} & x_1 - x_2 + x_3 - x_4 = 4, \\ & & x_1 + x_2 - x_3 - x_4 = 8, \\ & & x_j \geq 0, \quad j = 1, 2, 3, 4. \end{array}$$

Show that $(x_1,\, x_2,\, x_3,\, x_4) = (6,\, 2,\, 0,\, 0)$ is an optimal solution to P1. ... (3p)

(b) Assume that we change the equality constraint in P1 to an inequality constraint of the type $\geq$, so that the following LP-problem is obtained:

$$\begin{array}{lll} \text{P2:} & \text{minimize} & x_1 + 3x_2 + 2x_3 + x_4 \\ & \text{s.t.} & x_1 - x_2 + x_3 - x_4 \geq 4, \\ & & x_1 + x_2 - x_3 - x_4 \geq 8, \\ & & x_j \geq 0, \quad j = 1, 2, 3, 4. \end{array}$$

Determine, using the simplex method, an optimal solution to this problem P2. It is allowed to reuse the calculations made in the (a)-part above. .... (4p)

(c) Assume instead that we change the equality constraint in P1 to an inequality constraint of the type $\leq$, so that the following LP-problem is obtained:

$$\begin{array}{lll} \text{P3:} & \text{minimize} & x_1 + 3x_2 + 2x_3 + x_4 \\ & \text{s.t.} & x_1 - x_2 + x_3 - x_4 \leq 4, \\ & & x_1 + x_2 - x_3 - x_4 \leq 8, \\ & & x_j \geq 0, \quad j = 1, 2, 3, 4. \end{array}$$

Determine, using any method of choice, an optimal solution to this problem P3. .... (2p)

(d) Formulate the dual LP-problems corresponding to P1, P2 and P3 respectively. Determine the three different optimal solutions to these three dual problems. (3p)

3. Consider the following non-linear least-square problem in the variable vector $\mathbf{x} \in I\!R^2$:

$$\text{minimize} \quad f(\mathbf{x}) = \tfrac{1}{2}(h_1(\mathbf{x})^2 + h_2(\mathbf{x})^2 + h_3(\mathbf{x})^2 + h_4(\mathbf{x})^2), \quad \text{where}$$

$$\begin{aligned} h_1(\mathbf{x}) &= (x_1 + 2)^2 + x_2^2 - 5, \\ h_2(\mathbf{x}) &= (x_1 - 2)^2 + x_2^2 - 3, \\ h_3(\mathbf{x}) &= x_1^2 + (x_2 - 2)^2 - 6, \\ h_4(\mathbf{x}) &= x_1^2 + (x_2 + 2)^2 - 2. \end{aligned}$$

(a) Perform one iteration with Gauss-Newtons method starting from $\mathbf{x}^{(1)} = (0, 0)^{\mathsf{T}}$. Check that the point $\mathbf{x}^{(2)}$ you obtain satisfies $f(\mathbf{x}^{(2)}) < f(\mathbf{x}^{(1)})$. .... (6p)

(b) An alternative would of course be to use Newton's method to minimize $f(\mathbf{x})$. Show that for this particular problem, and from this specific starting point $\mathbf{x}^{(1)}$, the next iteration point $\mathbf{x}^{(2)}$ is the same for Newton's method as for Gauss-Newton's metod. .... (2p)

(c) Is the objective function $f(\mathbf{x})$ a convex function on the whole $I\!R^2$? Motivate your answer. .... (2p)

4. First recall that $\mathcal{N}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A})$ denotes the *nullspace* and the *range space* (also called the *column space*) of the matrix $\mathbf{A}$.

In this exercise $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$,

where b_1, b_2, c_1 and c_2 are given numbers.

(a) Determine the point among all points in $\mathcal{R}(\mathbf{A})$ which has the shortest distance to the given point $\mathbf{b}$. The answer may include b_1 and b_2. .... (2p)

(b) Determine the point among all points in $\mathcal{N}(\mathbf{A}^\mathsf{T})$ which has the shortest distance to the given point $\mathbf{b}$. The answer may include b_1 and b_2. .... (2p)

(c) Determine the point among all points in $\mathcal{R}(\mathbf{A}^\mathsf{T})$ which has the shortest distance to the given point $\mathbf{c}$. The answer may include c_1 and c_2. .... (2p)

(d) Determine the point among all points in $\mathcal{N}(\mathbf{A})$ which has the shortest distance to the given point $\mathbf{c}$. The answer may include c_1 and c_2. .... (2p)

5. In this exercise $\mathbf{H}$ is a given *positive definite* symmetric $n \times n$–matrix and $\mathbf{c}$ is a given vector $\neq \mathbf{0}$ in $\mathbb{R}^n$. Furthermore, f is a function from $\mathbb{R}^n$ to $\mathbb{R}$ definied by

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} + \mathbf{x}.$$

Let $\hat{f}_0$ denote the optimal value to the problem to minimize $f(\mathbf{x})$ without constraints.

(a) Consider first the following problem, where k_1 is a negative real constant:

$$\begin{aligned} \mathrm{P}_1: \quad \text{minimize} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \leq k_1. \end{aligned}$$

Let $\hat{f}_1$ denote the optimal value to this problem P_1.
Determine $\hat{f}_1$ expressed in $\hat{f}_0$ and k_1 (for $k_1 < 0$). .... (3p)

(b) Consider then the following problem, where k_2 is a positive real constant:

$$\begin{aligned} \mathrm{P}_2: \quad \text{minimize} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} \leq k_2. \end{aligned}$$

Let $\hat{f}_2$ denote the optimal value to this problem P_2.
Determine $\hat{f}_2$ expressed in $\hat{f}_0$ and k_2 (for $k_2 > 0$). .... (3p)

(c) Consider finally the following problem, where k_1 and k_2 are real constants with k_1 negative and k_2 positive:

$$\begin{aligned} \mathrm{P}_3: \quad \text{minimize} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \leq k_1, \\ & \mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} \leq k_2. \end{aligned}$$

Let $\hat{f}_3$ denote the optimal value to this problem P_3.
Determine $\hat{f}_3$ expressed in $\hat{f}_0$, k_1 and k_2 (for $k_1 < 0$ and $k_2 > 0$). .... (4p)

Good luck!