KTH Mathematics

Exam in SF1811/SF1831/SF1841 Optimization for F.
Saturday march 14 2009 kl. 08.00–13.00

Instructor: Per Enqvist, tel. 790 62 98

Allowed utensils: Pen, eraser and ruler. **No calculator!** A formula-sheet is handed out.

Solution methods: If not specifically stated in the problem statement, the problems should be solved using systematic methods that do not become futile for large problems. Unless so stated, known theorems can be used without proving them, assuming that they are stated correctly. Motivate your conclusions carefully.

A passing grade is guarranteed for 24 points, including bonus points from the voluntary home assignments. 21-23 points grant the possibility to complement the exam the result within three weeks from the announcement of the results. Contact the instructor if this is the case.

Write your name on each page of the solutions you hand in and number the pages. Write the solutions to the different questions 1,2,3,4,5 on separate sheets.

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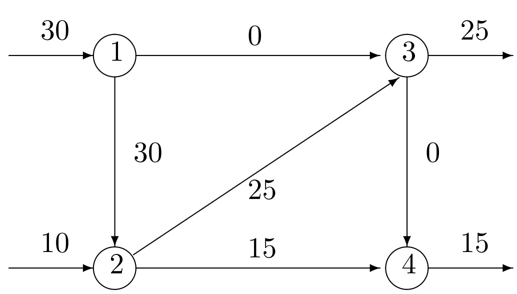
1. (a) Consider the following network



Show that the flow suggested in the graph does not minimize the cost of the flow if the costs in the links are given by

$$c_{12} = 1, c_{13} = 2, c_{23} = 2, c_{24} = 2, c_{34} = 2,$$

. . . . (3p)

(b) Determine the minimum cost flow in the network.

. . . . (4p)

(c) What defines a spanning tree in a graph ? Determine all the spanning trees in the graph above.

. . . . (2p)

2. (a) Consider the following linear programing problem

$$(P) \quad \left[\begin{array}{ll} \min\limits_{x} & x_1 - x_2 + 3x_3 \\ \text{s.t.} & x_1 - x_2 + 2x_3 = 1 \\ & 2x_1 + 3x_2 + x_3 \geq 7 \\ & x \geq 0 \end{array}\right].$$

Show that $x^{(a)} = (2, 1, 0)$ is optimal for (P).

. . . . (3p)

(b) If we modify the problem (P), to a new problem (P'), by changing the objective function so that $f(x) = 6x_1 - x_2 + 3x_3$, what is the optimal solution then ?
Hint: Write the modified version (P') of (P) on standard form and use the optimal solution from (a) as a starting basic solution. (Let x_1 and x_2 be the starting basic variables.)

. . . . (5p)

(c) Determine the dual optimization problem (D') to the modified problem (P') in (b). What is the optimal solution to the dual (D') ? . . . . (3p)

3. (a) Consider the unconstrained nonlinear minimization problem with the objective function

$$f(x) = \frac{1}{2}x^T Hx + c^T x + e^{x_1^2+x_2^2+x_3^2},$$

where

$$H = \begin{bmatrix} 0 & -2 & 4 \\ -2 & 3 & 2 \\ 4 & 2 & 14 \end{bmatrix}, \quad c^T = \begin{bmatrix} 2 & -2 & 4 \end{bmatrix}.$$

Show that the function is not convex by trying to factorize the Hessian of f at the origin on the form LDL^T. Motivate why this show that f is not convex.
Hint: To avoid some anoying derivation errors I tell you that the Hessian of the exponential part is diagonal

. . . . (4p)

(b) We know that the function will have a global minimum since the exponential term will become very large for large x. Newton's algorithm is not guarranteed to converge to the global optimum, but anyway, we would like to use Newton's method to determine the Newton direction at the origin.
Since a descent direction is guarranteed only if the Hessian is positive definite, the Hessian is usually modified to become positive definite if it is not true from the start.
Here, assume that the modified Hessian is given by

$$\tilde{H} = \begin{bmatrix} 2 & -2 & 4 \\ -2 & 5 & 2 \\ 4 & 2 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Motivate why $\tilde{H}$ is positive semidefinite, but not positive definite. . . . . (1p)

(c) Show that $-c \in \mathcal{R}(\tilde{H})$.

. . . . (2p)

(d) Show that the equation $\tilde{H}d = -c$ does not have a unique solution by showing that the nullspace of $\tilde{H}$ is one-dimensional.

. . . . (3p)

We leave the solving of the optimization problem for another time!

4. Consider the nonlinear programing problem

$$(P) \quad \left[\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s.t.} & x_1^2 + x_2^2 \le 1 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. \end{array}\right]$$

where $f(x) = \frac{1}{2}x^T H x + c^T x$,

$$H = \begin{bmatrix} -4 & 4 & 0 \\ 4 & 8 & 16 \\ 0 & 16 & 20 \end{bmatrix}, \qquad c = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

(a) Is any of the directions

$$d^{(1)} = (0,1,1)^T, \quad d^{(2)} = (0,1,-1)^T, \quad d^{(3)} = (1,1,1)^T$$

a feasible descent direction to f at the point $x^{(0)} = (0,0,0)$?
Motivate why each direction is a feasible descent direction or not.
Is the point $x^{(0)} = (0,0,0)$ a local minimum? .... (3p)

(b) Show that $x^{(1)} = (1,0,0)$ satisfies the first order KKT-conditions. .... (3p)

(c) Consider now the linearly constrained quadratic problem

$$(\tilde{P}) \quad \left[\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s.t.} & Ax = b \end{array}\right],$$

where $f(x)$ is defined above, and

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Use the nullspace method to determine an optimal solution, if it exists. . (3p)

(d) Is the point you determined in (c) a global optimum ? Motivate your answer. .... (1p)

5. Consider the nonlinear problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} \frac{a_i}{x_i} \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{m} \log x_i \leq b, \\ & x_i > 0, \quad i = 1, \cdots, m. \end{aligned}$$

where $a_i > 0$ for $i = 1, \cdots, m$.

(a) Use Lagrange relaxation to determine the dual problem. When you determine the dual, only the constraint on the sum of the $\log x_i$ should be relaxed, the positivity constraints should be kept implicit. .... (4p)

(b) Determine a stationary point to the dual problem. .... (2p)

(c) Use the solution in (b) to the dual problem to determine a solution to the primal problem .... (2p)

(d) Motivate global optimality for the primal and dual solutions. .... (2p)

Good luck!