KTH Mathematics

Exam in SF1811/SF1831/SF1841 Optimization for F.
Monday June 8 2009 kl. 14.00–19.00

Instructor: Per Enqvist, tel. 790 62 98

Allowed utensils: Pen, eraser and ruler. **No calculator!** A formula-sheet is handed out.

Solution methods: If not specifically stated in the problem statement, the problems should be solved using systematic methods that do not become futile for large problems. Unless so stated, known theorems can be used without proving them, assuming that they are stated correctly. Motivate your conclusions carefully.

A passing grade is guarranteed for 24 points, including bonus points from the voluntary home assignments. 21-23 points grant the possibility to complement the exam the result within three weeks from the announcement of the results. Contact the instructor if this is the case.

Write your name on each page of the solutions you hand in and number the pages. Write the solutions to the different questions 1,2,3,4,5 on separate sheets.

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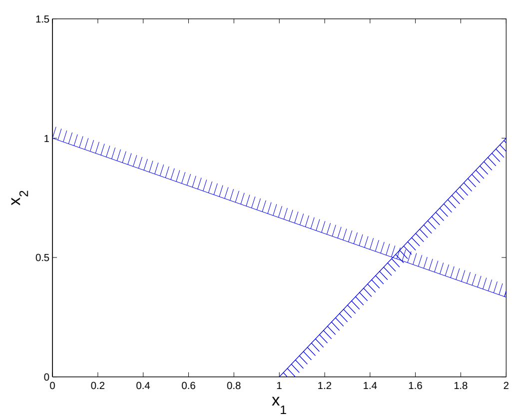
1. (a) Consider the linear programming problem

$$(P) \quad \begin{bmatrix} \min\limits_{x} & -x_1 - 2x_2 \\ \text{s.t.} & x_1 + 3x_2 \leq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{bmatrix},$$

The feasible region to (P) is depicted in the Figure below:



Write the problem (P) in standard form and solve the problem with the simplex method. Start with the slack-variables as basic variables. .... (5p)

(b) Determine the dual (D) of the original optimization problem (P). Find the optimal solution to the dual. .... (3p)

(c) Sometimes when you have equality constraints it is tempting solve for one variable and eliminate it from the other equations, but it is easy to make misstakes this way. Why does the problem

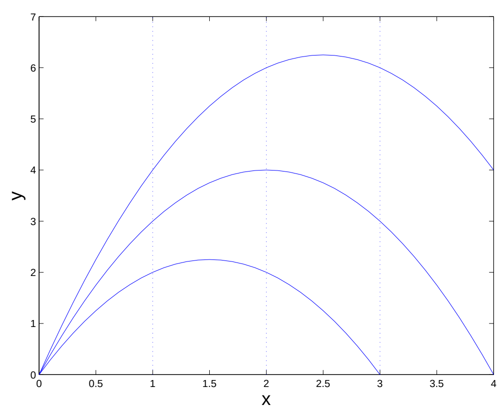
$$(P_1) \quad \left[\begin{array}{ll} \min\limits_{x} & x_1 + 2x_2 \\ \text{s.t.} & 2x_1 + x_2 = 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}\right],$$

and

$$(P_2) \quad \left[\begin{array}{ll} \min\limits_{x} & x_1 + 2(1 - 2x_1) \\ \text{s.t.} & x_1 \geq 0, \end{array}\right],$$

not have the same optimal values? .... (2p)

2. A projectile is launched from a position with the x-coordinate 0. The projectile has no propulsion system and does not have the ability to glide, so we can assume that its trajectory will describe a parabola where the height y depends on x as $y = \alpha x^2 + \beta x$ for some constants α and β.



At the distances x_k, $k = 1, \cdots, m$, measurements y_k of the height of the trajectory are made. These measurements will be used to estimate the coefficients α and β by minimizing the quadratic difference between the measurements and the function values.

(a) Formulate the problem above as a least squares problem, i.e. determine the matrices A and b and the vector of variables z so that the objective function is given by $\|Az - b\|^2$.
.... (2p)

(b) Assume that we at the distances 1,2 and 3 measured the heights 1, 4 and 2. What is the least squares estimation in this case for the equation of the trajectory ? .... (4p)

Comment: From physical considerations it should be assumed that the coefficient α is negative and β positive, but we neglect these constraints here.

The following formula can be useful

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

3. (a) Consider the quadratic minimization problem

$$\begin{array}{ll} \min\limits_{x} & \frac{1}{2}x^T H x + c^T x, \\ \text{s.t.} & Ax = b \end{array},$$

where

$$H = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}, \quad b = 2.$$

Show that the optimization problem is convex.
.... (2p)

(b) Let $\bar{x} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$. Since the problem in (a) is convex, and $\bar{x}$ is feasible, the following two statements are equivalent: There exists no feasible descent directions at $\bar{x}$ and that $H\bar{x} + c \in \mathcal{R}(A^T)$. (the range space of A^T)

Determine if there exists any feasible descent directions at $\bar{x}$.

Determine also if there exists any descent directions at $\bar{x}$. (not necessarily feasible)

Is $\bar{x}$ a global minimum for the problem in (a) ?

Motivate your answer well. .... (4p)

(c) Use LDL^T-factorization to determine if the matrices $H + 2I$ and $H + 3I$ are positive definite.

Does the corresponding two unconstrained optimization problems, *i.e.*

$$\min_{x} \ \tfrac{1}{2}x^T(H+2I)x, \ , \qquad \min_{x} \ \tfrac{1}{2}x^T(H+3I)x, \ ,$$

respectively, have a finite optimal solution ?

What is the finite optimal solution if it exists ? .... (4p)

4. Consider the nonlinear programing problem

$$(P) \quad \left[\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s.t.} & x_1^2 + x_2^2 + x_3^2 \leq 1 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{array}\right]$$

where $f(x) = (x_1 - 1/2)(x_2 - 1/3)(x_3 - 1/4)$.

(a) Assume that none of the constraints are active and determine 3 different points that satisfies the first order necessary conditions for a local minimum. ...(3p)

(b) Does the point $x^{(0)} = (1/2, 1/3, 1/4)$ satisfy the second order necessary conditions for a local minimum ?
Is it a local minimum ? ....(4p)

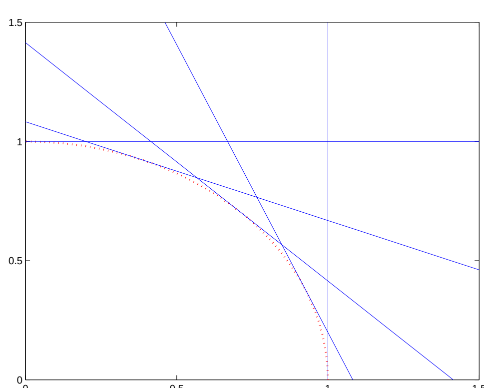
(c) Does the point $x^{(1)} = (0, 0, 1)$ satisfy the KKT optimality conditions ?
Is the answer enough to decide if $x^{(1)}$ is, or is not, a local minimum ? ...(2p)

(d) Does the point $x^{(2)} = (0, 0, 0)$ satisfy the KKT optimality conditions ?
Is the answer enough to decide if $x^{(2)}$ is, or is not, a local minimum ? ...(2p)

5. Consider the nonlinear problem:

$$\begin{array}{rl} \text{minimize} & -x_1 - 2x_2 \\ \text{s.t.} & x_1^2 + x_2^2 \leq 1, \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

Assume that we want to approximate the nonlinear constraints using linear constraints. Use the tangents of the constraint at the points $(x_1^{(k)}, x_2^{(k)}) = (\cos(v_k), \sin(v_k))$ for $v_0, v_1, \cdots, v_N$ given by $v_k = k\pi/N$.



(a) Formulate the linear approximative optimization problem in standard form that corresponds to the nonlinear optimization problem above. ....(3p)

(b) How are the optimal values of the nonlinear and linear optimization problems in (a) related ? ....(1p)

(c) The dual of the original nonlinear optimization problem is given by

$$\begin{aligned} \text{maximize} \quad & \varphi(\lambda) = -\frac{5}{4}\frac{1}{\lambda} - \lambda \\ \text{s.t.} \quad & \lambda > 0. \end{aligned}$$

Show this. .... (4p)

(d) The dual optimization problem in (c) can be approximated by a linear optimization problem by making a piecewise linear approximation of the dual objective function, namely the lower envelope of the tangents of the graph of φ at some fixed positive values $\lambda_0, \lambda_1, \cdots, \lambda_N$. That is, for a fixed λ we approximate $\varphi(\lambda)$ with the value of the tangent that lies below the others for that λ.

Formulate this as a linear optimization problem .... (4p)

(e) How are the optimal values of the nonlinear dual and the linear optimization problem in (d) related ? .... (1p)

Motivate your answers well!

Good luck!