KTH Mathematics

Exam in SF1811/SF1831/SF1841 Optimization for F.
Tuesday August 25, 2009, kl. 14.00–19.00

Instructor: Per Enqvist, tel. 790 62 98

Allowed utensils: Pen, eraser and ruler. **No calculator!** A formula-sheet is handed out.

Solution methods: If not specifically stated in the problem statement, the problems should be solved using systematic methods that do not become futile for large problems. Unless so stated, known theorems can be used without proving them, assuming that they are stated correctly. Motivate your conclusions carefully.

A passing grade is guarranteed for 25 points, including bonus points from the voluntary home assignments. 23-24 points grant the possibility to complement the exam within three weeks from the announcement of the results. Contact the instructor if this is the case.

Write your name on each page of the solutions you hand in and number the pages. Write the solutions to the different questions 1,2,3,4,5 on separate sheets.

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1. (a) Consider the following distribution planning problem.

The nodes 1 and 2 are source nodes with net outflow of 10 and 20 fully loaded lorries per hour respectively. The nodes 4 and 5 are sink nodes with the net inflow of 15 lorries each. The traffic can go from node 1 to 4 and from node 2 to 5. It can also go from node 1 as well as 2 to the transshipment node 3 and then further on to the nodes 4 as well as 5.

The traffic intensity is assumed to be so low that we can disregard all traffic congestion issues. We also consider the lorry cargos as replaceable units, i.e. it does not matter which of the lorries that starts in the source nodes and which sink node they arrive at.

To each arc in the graph is an associated cost. Give an example of a reasonable choice of costs for a real traffic problem. (That is, what these costs conceptually corresponds to)

Draw a graph describing the network. Determine in an arbitrary way a feasible basic solution that corresponds to a spanning tree in the network graph.

Assume that the costs are 2 units/lorry for each of the arcs. Determine the optimal solution to the problem.

Hint: The easiest way to do this is to start with the solution you think is optimal and then to verify this using reduced costs. It is not a systematic approach, but we allow it in this problem. . . . . (5p)

(b) Write the linear optimization problem

$$(P)\quad \left[\begin{array}{ll} \min\limits_{x} & x_1 - x_2 + 3x_3 \\ \text{s.t.} & x_1 + x_2 - x_3 \leq 1 \\ & 2x_1 + 3x_2 - x_3 \geq 2 \\ & x_2 \geq 0, x_3 \geq 0. \end{array}\right],$$

in standard form. . . . . (2p)

(c) Assume that x is a feasible point to the optimization problem

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

and that y is a feasible point to the dual optimization problem

$$\begin{array}{rl} \text{maximize} & b^T y \\ \text{s.t.} & A^T y \leq c. \end{array}$$

Show that $c^T x \geq b^T y$ must hold in this case. .... (2p)

2. (a) Consider the following linear optimization problem:

$$(P) \quad \left[\begin{array}{ll} \min\limits_{x} & x_1 - 2x_2 \\ \text{s.t.} & x_1 + 3x_2 \leq 1 \\ & 2x_1 - x_2 \leq 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{array} \right].$$

Write the problem on standard form with the matrices (A, b, c) and solve it with the simplex algorithm.
Start with the variables $\{1, 2\}$ as basic variables. .... (4p)

(b) Determine a nullspace matrix Z to the A-matrix corresponding to the constraints in (a) on standard form.

Let $\bar{x}$ be the starting basic solution used in (a), and show that the optimal solution $\hat{x}$ can be written $\hat{x} = \bar{x} + Zv$ for some vector v. Determine the vector v.

If you did not solve the (a)-problem you may use $\hat{x} = (0\ 0\ 1\ 1)^T$ instead. It is not optimal, but you should still be able to solve this part of the problem. (3p)

(c) Determine if the c-vector in the (a)-part is in the range space of the A^T-matrix. .... (2p)

(d) Determine if the c-vector in the (a)-part is in the orthogonal complement subspace to the range space of the A^T-matrix. .... (2p)

3. Consider the quadratic optimization problem

$$(P) \quad \left[\begin{array}{ll} \text{minimera} & f(x) \\ \text{s.t.} & Ax = b \end{array} \right],$$

where $f(x) = \frac{1}{2} x^T H x + c^T x$ and A, b, H, c are given by

$$A = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 10 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -1 \\ 4 \end{bmatrix}.$$

(a) Show that f is a strictly convex function on the whole $\mathbb{R}^3$. . . . . (2p)

(b) Use the nullspace method to determine the minimum of the problem. Is this a global minimum ? . . . . (5p)

(c) Verify that the optimal point you determined in the (b)-part solves the equation system forming the Lagrange conditions, and determine the corresponding Lagrange multipliers. . . . . (3p)

4. Consider the non-linear optimization problem

$$(P) \quad \left[\begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t.} & x_1^2 + x_2^2 + x_3^2 = 1 \\ & x_1^2 - 1 = x_2 \\ & x \in \mathbb{R}^3 \end{array}\right]$$

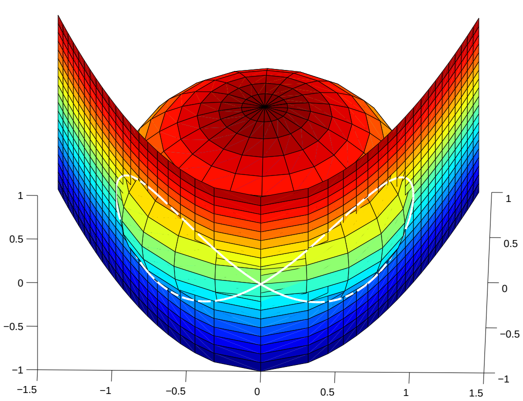
where $f(x) = e^{x_1+x_3} + x_2^4 + 4x_2 - (x_1 + x_3)$,

(a) Is the objective function convex ?
Is the feasible region convex ?
Is this a convex optimization problem ?
Motivate your answers. . . . . (3p)

(b) There are three feasible points such that $x_3 = 0$, find them. . . . . (1p)

(c) Which of the points in (b) satisfies the KKT-conditions ? . . . . (3p)

(d) Is any of the points in (b) globally optimal for the optimization problem ? Motivate carefully. . . . . (3p)



5. Consider the non-linear optimization problem

$$(P) \quad \left[\begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t.} & x_1^2 + x_2^2 \leq 1 \\ & x_1^2 - 1 \leq x_2 \\ & x \in \mathbb{R}^2 \end{array}\right]$$

where $f(x) = e^{-x_1} - x_2^2$,

(a) Is the objective function convex ?
Is the feasible region convex ?
Is this a convex optimization problem ?
Motivate your answers. .... (2p)

(b) Does the point $x^{(b)} = (1, 0)$ satisfy the KKT-conditions ?
Is it based on this result possible to say that $x^{(b)}$ is a local minimum ?
Motivate your answer well. .... (3p)

(c) It can e shown that the global optimum lies somewhere along the arc of the circle that lies in the positive orthant. Which equation must the x_1 coordinate for this point satisfy? .... (3p)

(d) Show that the point in (c) in fact is a global optimum. .... (2p)

Motivate your answers well!

Good luck!