KTH Mathematics

Exam in SF1811/SF1831/SF1841 Optimization.
Wednesday March 14, 2012, time: 14.00–19.00

Examiner: Per Enqvist, tel. 790 62 98

Allowed utensils: Pen, paper, eraser and ruler. **No calculator!** A formula-sheet is handed out.

Solution methods: If not specifically stated in the problem statement, the problems should be solved using systematic methods that do not become futile for large problems. Unless so stated, known theorems can be used without proving them, assuming that they are stated correctly. Motivate your conclusions carefully.

A passing grade is guarranteed for 25 points, including bonus points from the voluntary home assignments. 23-24 points grant the possibility to complement the exam within three weeks from the announcement of the results. Contact the instructor asap if this is the case.

Write your name on each page of the solutions you hand in and number the pages. Write the solutions to the different questions 1,2,3,4,5 on separate sheets. (This is important since the exams are split up for fair correction)

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1. (a) There is a party and you are going to plan the seating for your guests in an optimal way. You have invited 18 guests and there are three tables with six chairs at each table.

Formulate the optimization problem for maximizing the total utility function. The utility for person i is u_{ik} (given numbers) if seated at table k and the total utility is the sum of all the persons utilities.

The utilities are individual and reflect each persons priorities between factors, such as, prestige, accessibility, and comfort. We assume that the utilities are fixed and do not depend on the other persons choices.

Recommend how this problem should be solved. .... (4p)

(b) Consider the problem

$$(LP)\qquad \begin{array}{rl} \text{minimize} & -2x_1 + x_2 \\ \text{s.t.} & x_1 - x_2 \leq 0 \\ & x_2 \leq 1 \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

Let $x^* = (0, 0)$, and determine
all feasible directions at x^*
all descent directions at x^*
all feasible descent directions at x^*
what is the optimal point $\hat{x}$ of (LP) and does there exist a feasible descent direction d at x^* pointing to $\hat{x}$.
You may determine this graphically, but explain your figures well. .... (4p)

(c) Assume that x is a feasible point to the optimization problem

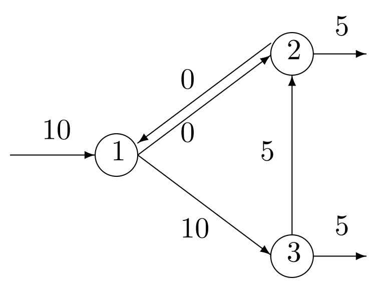
$$(P) \qquad \begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

and that y is a feasible point to the dual optimization problem

$$(D) \qquad \begin{aligned} \text{maximize} \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c. \end{aligned}$$

Show that $c^T x \geq b^T y$ must hold in this case. .... (2p)

2. Consider the following network



Assume that the cost of the flow in the links are given by

$$c_{12} = 2, c_{13} = 1, c_{21} = -1, c_{32} = 2,$$

and that we want to minimize the total cost of a feasible flow in the network.

(a) Determine the adjacency matrix A and vectors b and c such that the minimum cost flow problem becomes a linear program on standard form. Find a formulation where there are as few linear constraints as possible. .... (2p)

(b) Use the simplex method, for general linear programs on standard form or the network version, to find out if the flow suggested in the graph above is optimal, and if not determine the minimizing flow or prove that there is no minimizing flow. .... (3p)

(c) Determine the dual to the minimum cost flow problem and find the optimal solution to the dual. Write it out explicitly, i.e. we want to see each constraint written out on component form; no general matrix expression.

If you did not solve (b) you may determine the optimal solution to (b) by inspection and use it to determine the optimal solution to the dual, but it will not give you any points on part (b).

Show that the complementarity conditions are satisfied for the solutions to the primal and dual problems. .... (5p)

3. Consider the quadratic optimization problem

$$(P) \quad \left[\begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t.} & Ax = b \end{array}\right],$$

where $f(x) = \frac{1}{2}x^T Hx + c^T x$ and A, b, H, c are given by

$$A = \left[\begin{array}{ccc} 2 & -4 & 2 \\ 6 & 0 & 2 \end{array}\right], \quad b = \left[\begin{array}{c} 1 \\ 1 \end{array}\right], \quad H = \left[\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 4 & 4 \end{array}\right], \quad c = \left[\begin{array}{c} 3 \\ 3 \\ 0 \end{array}\right].$$

(a) Is f a convex function on the whole $\mathbb{R}^3$. ....(2p)

(b) Use the nullspace method to determine if there exists a minimum to the problem, and if it does determine the minimizing x^*. .... (5p)

(c) Show that $\nabla f(x^*)$ is linearly dependent of the rows in A. Is it a coincidence that this holds?
If you found no minimum in (b), you may use $x^* = (1, 1, 1)^T$ to solve part (c) of the exercise. .... (3p)

4. Consider the non-linear optimization problem

$$(NLP) \quad \left[\begin{array}{ll} \text{minimize} & -2x + 2y - z \\ \text{s.t.} & x^2 + 4y^2 \leq 1 \\ & x + z \leq 0 \\ & xyz \leq 0 \\ & x, y, z \in \mathbb{R} \end{array}\right]$$

(a) Is the objective function convex ? Is the feasible region convex ? Is this a convex optimization problem ? Motivate your answers. ....(3p)

(b) Are there any points that satisfy the KKT conditions when only the first two constraints are active, i.e. satisfied with equality.
If there is such a point, motivate well if it is a local minimum or if you even can say that it is a global minimum. ....(7p)

5. Consider the non-linear optimization problem

$$(P) \quad \left[\begin{array}{ll} \displaystyle\min_{p_1,\cdots,p_n} & \displaystyle\sum_{i=1}^{n} p_i \log p_i \\ \text{s.t.} & \displaystyle\sum_{i=1}^{n} p_i \leq 1 \\ & \displaystyle\sum_{i=1}^{n} ip_i \leq \mu \\ & \displaystyle\sum_{i=1}^{n} (i-\mu)^2 p_i \leq \sigma^2 \\ & p_i \geq 0, \quad i = 1, \cdots, n. \end{array}\right]$$

A short note on interpretations of (P):
If we had linear equality constraints and interpreted p_i as the probability for the event $X = i$, then the first condition would say that the sum of the probabilities is one, the second that the mean of the variables X is μ and the third that the variance of the variables X is σ^2. But here we will consider the optimization problem with inequalities as described above. The objective function with negative sign is called the entropy of the probability distribution, so (P) corresponds to an entropy maximization problem.

The feasible set for this problem might very well be empty, e.g. if μ is smaller than 1, then there will exist no feasible solutions. So the choice of the parameters μ and σ are important.

(a) Assume that we know that the parameters μ and σ are such that there exists an open subset of $\mathbb{R}^n_+$ of feasible points to (P). Is the problem (P) a regular convex optimization problem ? ....(3p)

(b) Formulate the Lagrange function for the problem and determine the dual optimization problem. Determine an explicit expression for the optimal probablities p_i in terms of the dual variables. ....(5p)

(c) Assume that there is a point $\hat{y}$ that is feasible for the dual and such that the gradient of the dual objective function is zero, what can we then say about the solution to the primal problem? ....(2p)

Motivate your answers well!

Good luck!