KTH Mathematics

Exam in SF1811/SF1841 Optimization.
Monday March 13, 2013, time: 8.00–13.00

Examiner: Per Enqvist, tel. 790 62 98

Allowed utensils: Pen, paper, eraser and ruler. **No calculator!** A formula-sheet is handed out.

Solution methods: If not specifically stated in the problem statement, the problems should be solved using systematic methods that do not become futile for large problems. Unless so stated, known theorems can be used without proving them, assuming that they are stated correctly. Motivate your conclusions carefully.

A passing grade is guarranteed for 25 points, including bonus points from the voluntary home assignments. 23-24 points grant the possibility to complement the exam within three weeks from the announcement of the results. Contact the instructor asap if this is the case.

Write your name on each page of the solutions you hand in and number the pages. Write the solutions to the different questions 1,2,3,4,5 on separate sheets. (This is important since the exams are split up and corrected separately for fair correction)

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1. Consider a network flow problem with four nodes, numbered 1 to 4. There is a directed arc from node i to j in the network if $i < j$, and the cost per unit of flow in that link is $8 - i - j$. (the flow in the network is only allowed in the directions of the arcs) There is a net flow in to node 1 of 6 units and a net flow out of nodes 2,3 and 4 of 2,3 and 1 units correspondingly.

(a) Solve the minimum cost flow problem using the (network version) of the simplex method starting by only allowing flows in the arcs $(1, 2)$, $(2, 3)$ and $(3, 4)$. (6p)

(b) Assume that A is the adjacency matrix of the network described above, and let d be a vector in the nullspace of A.

Under what conditions on the current flow x in the network is d then a feasible direction, for the flow problem described above, at the flow x.

Under what further conditions is d also a feasible descent direction?

Does there exist any feasible descent directions for the initial flow used in part (a)? .... (3p)

2. (a) Consider the following linear optimization problem:

$$(P) \quad \left[\begin{array}{ll} \max\limits_{\mathbf{x}} & 2x_1 + 3x_2 + 3x_3 \\ \text{s.t.} & x_1 + x_2 + 2x_3 \leq 3 \\ & x_1 + 2x_2 + x_3 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{array} \right].$$

Write the problem on standard form with the matrices (A, b, c) and solve it with the simplex algorithm. ....(5p)

(b) Determine explicitly the dual (D) of (P). *Do not determine the dual to the problem on standard form.* (no matrices in the answer) ....(2p)

(c) Determine the optimal solution to the dual using a graphical approach.

Show that the complementarity conditions are satisfied for the optimal solutions to the primal and dual problems. ....(4p)

3. Consider the quadratic optimization problem

$$(P) \quad \left[\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{s.t.} & A\mathbf{x} = b \end{array} \right],$$

where $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T H \mathbf{x} + c^T \mathbf{x}$ and A, b, H, c are given by

$$A = \left[\begin{array}{ccc} 1 & 1 & 1 \end{array} \right], \quad b = \left[\begin{array}{c} 1 \end{array} \right], \quad H = \left[\begin{array}{ccc} 1 & 2 & 1 \\ 2 & 6 & 6 \\ 1 & 6 & 10 \end{array} \right], \quad c = \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right].$$

(a) Is f a convex function on the whole $\mathbb{R}^3$. ....(2p)

(b) Use the nullspace method to determine if there exists a minimum to the problem, and if it does determine the minimizing $\mathbf{x}^*$. ....(5p)

(c) Solve the problem using the Lagrange method. ....(3p)

4. Let $f(x, y) = x \log(x/y) + y/e$, where $\log(\cdot)$ denotes the natural logarithm in the base e, and we assume that $x \geq 0.1$ and $y \geq 0.1$.

(a) Determine all feasible points (x, y) that satisfies the first order optimality conditions.

Determine if the second order necessary or sufficient conditions for a minimum are satisfied for those points.

What can we say about the optimality of those points based on what we have determined so far?

Is it possible to make a stronger statement about the optimality using some other argument? .... (5p)

(b) Assume that we add a further constraint, namely $h(x, y) = x + y - 1 = 0$. Determine all points that satisfy the Lagrange optimality conditions and make an as strong as possible statement about the optimality of those points. . (4p)

5. Consider the nonlinear optimization problem

$$(P) \quad \begin{bmatrix} \text{minimize}_{\mathbf{x} \in \mathbb{R}^n} & f_0(\mathbf{x}) \\ \text{s.t.} & f_i(\mathbf{x}) \leq 0, \quad i = 1, \cdots, m. \end{bmatrix}$$

We assume that it is a convex optimization problem.

Denote the feasible region by

$$\mathcal{F} = \{\mathbf{x} \in \mathbb{R}^n \; : \; f_i(\mathbf{x}) \leq 0, \quad i = 1, \cdots, m\}$$

(a) The epigraph of a function can geometrically be defined as the set of points that lies above, or on, the graph of the function. Show that the epigraph of f_0 is convex, *i.e.*, that the set

$$\text{epi}(f_0) = \{(t, \mathbf{x}) \in \mathbb{R} \times \mathcal{F} \; : \; f_0(\mathbf{x}) \leq t\}$$

is convex. .... (3p)

(b) Show that if $(t, \mathbf{x})$ is a point that satisfies the KKT-conditions for the problem (with linear objective function)

$$(P_\ell) \quad \begin{bmatrix} \text{minimize}_{\mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}} & t \\ \text{s.t.} & f_0(\mathbf{x}) - t \leq 0, \\ & f_i(\mathbf{x}) \leq 0, \quad i = 1, \cdots, m, \end{bmatrix}$$

then the point $\mathbf{x}$ satisfies the KKT-conditions for the problem (P). ...... (3p)

(c) Assume that the objective function is given by $f_0(\mathbf{x}) = x_1$, and the feasible region is determined by $m = 2$, $f_1(\mathbf{x}) = x_2^2 + x_3^2 - x_1$ and $f_2(\mathbf{x}) = 1 - (x_2 + x_3)$. Are there any $\mathbf{x}$ that satisfies the KKT-conditions, and what can we say about the optimality of those points? .... (5p)

Good Luck!