KTH Mathematics

Exam in SF1811/SF1841 Optimization.
Wednesday May 29, 2013, time: 8.00–13.00

Examiner: Per Enqvist, tel. 790 62 98

Allowed utensils: Pen, paper, eraser and ruler. **No calculator!** A formula-sheet is handed out.

Solution methods: If not specifically stated in the problem statement, the problems should be solved using systematic methods that do not become futile for large problems. Unless so stated, known theorems can be used without proving them, assuming that they are stated correctly. Motivate your conclusions carefully.

A passing grade is guarranteed for 25 points, including bonus points from the voluntary home assignments. 23-24 points grant the possibility to complement the exam within three weeks from the announcement of the results. Contact the instructor asap if this is the case.

Write your name on each page of the solutions you hand in and number the pages. Write the solutions to the different questions 1,2,3,4,5 on separate sheets. (This is important since the exams are split up and corrected separately for fair correction)

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1. Consider a transportation problem with three factories (sources), and three warehouses (sinks). The supplies are 4,4 and 12, and the demands are 7, 8 and 5 units. Assume that all sources are connected to all sinks, and that the transportation cost c_{ij} per unit transported from source i to sink j is given by the matrix

c_{ij}	$j=1$	$j=2$	$j=3$
$i=1$	2	3	2
$i=2$	1	2	2
$i=3$	2	1	2

(a) Find a basic feasible solution to start with using the North-West corner rule. Use the specialized simplex method method to solve the problem from that initial bfs.

(You may use the network simplex method also, but you should not use the standard simplex method) .... (6p)

(b) The adjacency matrix of the transportation problem is given by A;

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}.$$

(it may look different depending on how you order the equations and the variables)

Determine the range space and the null space of A.

Use Gauss-Jordans method. You may also use the interpretations of the subspaces described in class, but then you have to motivate why you get the right number of basis vectors. .... (4p)

2. (a) Consider the following linear optimization problem:

$$(P) \quad \left[\begin{array}{ll} \max\limits_{\mathbf{x}} & 2x_1 + 3x_2 + 3x_3 - x_4 - x_5 + 2x_6 \\ \text{s.t.} & x_1 + x_2 + x_3 + 2x_5 + x_6 = 3 \\ & x_1 - x_2 + x_4 + 2x_6 = 3 \\ & x_k \geq 0, \quad k = 1, \cdots, 6. \end{array} \right].$$

Solve (P) with the simplex algorithm and start with variables x_1 and x_3 as basic variables. .... (5p)

(b) Determine explicitly the dual (D) of (P). .... (2p)

(c) Consider the simplex multiplicators of the starting bfs in (a) and of the optimal bfs. Use them as values for the dual variables for the dual (D) in (b) and determine the objective function values of the dual at each point, and check if both points are feasible.

Is this result in accordance with the weak duality theorem? .... (3p)

3. We want to fit a polynomial to some given data points.

Let $\Psi(t) = \sum_{k=1}^{3} \alpha_k \psi_k(t)$, where $\psi_1(t) = 1$, $\psi_2(t) = 1 + t$, and $\psi_3(t) = 1 + t + t^2$.

(a) Find the coefficients α_k that minimizes the quadratic distance between $(\Psi(-1), \Psi(1))$ and $(0, 1)$, and if there is not a unique solution find the one with the smallest 2-norm (of the coefficients α_k). .... (4p)

(b) Find the coefficients α_k that minimizes the quadratic distance between $(\Psi(-1), \Psi(1))$ and $(0, 1)$, under the additional constraint that the sum of the coefficients $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

How does the optimal objective function value vary from (a) to (b)? Interpretation? .... (6p)

4. Let $f(x, y) = xy + \frac{1}{3}\left(x^3 + y^3\right)$.

(a) Determine all spoints (x, y) that satisfies the first order optimality conditions.

Determine if the second order necessary or sufficient conditions for a minimum are satisfied for those points.

What can we say about the optimality of those points based on the first and second order conditions?

What can we say about global optimality for the problem? .... (5p)

(b) Assume that we add a further constraint, namely $h(x, y) = x + y - 1 = 0$. Determine all points that satisfy the Lagrange optimality conditions and make an as strong as possible statement about the optimality of those points. . (5p)

5. Consider the nonlinear optimization problem

$$(P) \quad \left[\begin{array}{lll} \text{minimize}_{\mathbf{x}\in\mathbb{R}^n} & f_0(\mathbf{x}) & \\ \text{s.t.} & f_i(\mathbf{x}) \leq 0, & i = 1, \cdots, n+1. \end{array} \right]$$

where

$$f_0(x) = \sum_{k=1}^{n} k e^{-x_k}$$

$$f_k(x) = x_k - 1, \quad k = 1, \cdots, n,$$

$$f_{n+1}(x) = \sum_{k=1}^{n} x_k - m.$$

(a) Is (P) a convex optimization problem? .... (2p)

(b) Let $n = 3$ and $m = 2.5$. Determine if there are any points that satisfies the KKT conditions, and if so, find them and make an as strong statement about optimality as possible. .... (5p)

(c) For the general problem, *i.e.*, arbitrary n; what can you say about the optimal solution and how it depends on m ? .... (3p)

You may use the following approximative values for the exponential function.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$\exp(x)$	0.37	0.61	1.00	1.65	2.72

Good Luck!