KTH Mathematics

Solutions for the exam in SF1841 Optimization for F.
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There may be alternative solutions to the problem.

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1. (a) Define variables x_A, x_B, x_C, x_D for how many manhours the respective persons are spending in the kitchen, y_A, y_B, y_C, y_D for how many manhours they are working at the register, and z_A, z_B, z_C, z_D for how many manhours they are cleaning.

The cost for the restaurant is given by the objective function

$$f = (x_A+y_A+z_A)*70+(x_A+y_A+z_A)*80+(x_A+y_A+z_A)*100+(x_A+y_A+z_A)*110+y_B*40$$

The last term is the cost of Bobbys money-problems.

All the work at El Burgo has to be done:

$$x_A + x_B + x_C + x_D = 40,$$

$$y_A + y_B + y_C + y_D = 60,$$

$$z_A + z_B + z_C + z_D = 40.$$

All working times has to be non-negative:

$$x_A \geq 0, x_B \geq 0, x_C \geq 0, x_D \geq 0$$

$$y_A \geq 0, y_B \geq 0, y_C \geq 0, y_D \geq 0$$

$$z_A \geq 0, z_B \geq 0, z_C \geq 0, z_D \geq 0$$

Furthermore, $x_C = 0$ and $x_D \geq 20$ are the constraints regulating the work of Cecile and Davide in the kitchen. These constraints make the constraints $x_C \geq 0$ and $x_D \geq 0$ above redundant, but the formulation as a linear program is ok with our without the redundant constraints. None are allowed to work more than 40 manhours per week:

$$x_A + y_A + z_A \leq 40, \quad x_B + y_B + z_B \leq 40,$$

$$x_C + y_C + z_C \leq 40, \quad x_D + y_D + z_D \leq 40.$$

Finally, to make sure that Adrian is not upset we require that $z_A \leq z_B$ and $z_A \leq z_C$

(b) Setting the node potential $v_5 = 0$, and using that $c_{ij} = v_i - v_j$ for the ij corresponding to basic variables, then the rest of the node potentials are given by

$$v_1 = 3, v_2 = 2, v_3 = 1, v_4 = 1.$$

Then the reduced costs for the other indices are given by $r_{ij} = c_{ij} - v_i + v_j$:

$$r_{12} = 2-3+2 = 1, r_{25} = 3-2+0 = 1, r_{34} = 1-1+1 = 1, r_{35} = 2-1+0 = 1.$$

Since all the reduced costs are non-negative the proposed solution is optimal.

(c) If $c_{25} = 1$, then the node potentials determined above are unchanged. The only thing that changes is the reduced cost r_{25} which is now given by $r_{25} = 1-2+0 = -1$. Since this reduced cost is negative, the value of the objective function decreases if x_{25} becomes a basic variable and increases from zero. The basic solution was non-degenarate so this implies that x_{25} can be increased and then the same solution can not be optimal.

2. (a) The starting basic solution is given by $x_1 = 5/3$, $x_2 = 2/3$, and the non-basic variables are $x_3 = x_4 = 0$. The reduced cost of x_3 is $r_3 = -2/3$. Letting x_3 enter the basis and x_2 exit, the new basic solution is $x_1 = 1$, $x_3 = 2$ and $x_2 = x_4 = 0$. The reduced costs are then all positive so the solution is optimal. The value of the objective function is then $3x_1 + x_3 = 3 + 2 = 5$.

(b) The dual linear programing problem is

$$(D) \quad \left[\begin{array}{ll} \max\limits_{y} & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \text{ free} \end{array} \right]$$

(c) By complementarity we know that $\hat{x}_i (A^T y - c)_i = 0$. Therefore, the first and third constraints in the dual are active:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

This gives the solution $\lambda_1 = 2$ and $\lambda_2 = -1$.

The optimal value of the dual is then $4 * 2 + 3 * (-1) = 5$, which is the same as for the primal. The two points then have to be optimal for the respective problems.

3. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(a) The range space of A is $\mathbb{R}^2$ which has dimension 2 and a basis is given by, for example, e_1, e_2 the columns of the identity matrix of dimension 2.
The null space of A is spanned by the columns of the matrix

$$Z = \begin{bmatrix} 0 & 2 \\ -1/2 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

which also form a basis for the space. Clearly the dimension is 2.
The range space of A^T is spanned by the columns of the matrix

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1/2 \\ -2 & 2 \end{bmatrix}.$$

which also form a basis for the space. Clearly the dimension is 2.
The null space of A^T is trivial, i.e. only the point $(0,0)$ which has dimension 0.

(b) The reduced Hessian is given by

$$Z^T H Z = \begin{bmatrix} 1/4 & 1 \\ 1 & 7 \end{bmatrix}.$$

Let $\bar{x} = (1,1,1,1)^T$, then

$$Z^T H \bar{x} = \begin{bmatrix} -1/2 \\ -1 \end{bmatrix}.$$

Solving $Z^T H Z v = -Z^T H \bar{x}$ gives $v = (10/3, -1/3)^T$, and then $\hat{x} = \bar{x} + Zv = (1, 0, 13, 2)^T/3$.

(c) The minimal value of the problem can not decrease if we add another constraint to the poblem, and since $\hat{x}$ is still feasible and the objective function is the same it has to be optimal also for the new problem.

4. Let

$$f(x) = x_1^2 - 4x_1x_2 + 2x_2^2 - 2x_2x_3 - 2x_2 + 3x_3$$
$$g_1(x) = 2 - (x_1^2 + x_2^2 + x_3^2)$$
$$g_2(x) = x_1$$
$$g_3(x) = x_2$$

$$g_4(x) = x_3$$

Then, the problem can be written $\min f(x)$ subject to $g_i(x) \geq 0$ for $i = 1, 2, 3, 4$.

Then

$$\nabla f(x)^T = \begin{pmatrix} 2x_1 - 4x_2 \\ -4x_1 + 4x_2 - 2x_3 - 2 \\ -2x_2 + 2x_3 + 3 \end{pmatrix}, \quad \nabla g_1(x)^T = -\begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$$\nabla g_2(x)^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \nabla g_3(x)^T = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \nabla g_4(x)^T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) At $x^{(0)} = (0, 0, 0)$ constraints 2,3,4 are active. For the KKT-conditions to be satisfied we need to find non-negative Lagrange parameters such that

$$\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \lambda_2^{(0)} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \lambda_3^{(0)} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \lambda_4^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

But here $\lambda_3^{(0)} = -2$, so the KKT-conditions are not satisfied.

(b) The point $x = (1, 1, 0)$ is feasible and the constraints 1 and 4 are active. For the KKT-conditions to be satisfied we need to find non-negative Lagrange parameters such that

$$\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} \lambda_1^{(1)} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \lambda_4^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This holds for $\lambda_1^{(1)} = 1$ and $\lambda_4^{(1)} = 1$, which are positive. Then the KKT-conditions are all satisfied.

(c) Let

$$Q(x) = x_1^2 - 4x_1x_2 + 6x_2^2 - 8x_2.$$

Then

$$\nabla Q(x)^T = \begin{pmatrix} 2x_1 - 4x_2 \\ -4x_1 + 12x_2 - 8 \end{pmatrix}, \quad \nabla^2 Q(x) = \begin{pmatrix} 2 & -4 \\ -4 & 12 \end{pmatrix}.$$

Starting at $(0, 0)$, the Newton direction d is given by the solution

$$\begin{pmatrix} 2 & -4 \\ -4 & 12 \end{pmatrix} d = -\begin{pmatrix} 0 \\ -8 \end{pmatrix}, \qquad d = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

The new point determined by taking the unit Newton step is $x = (4, 2)$. At the starting point the objective function value is zero, and at the new point it is $4^2 - 4 * 4 * 2 + 6 * 2^2 - 8 * 2 = -8$.

(d) Newtons method always gives the optimal solution in one step for convex quadratic functions. The objective function is convex and it is easy to check that the derivative ∇Q is zero at the new point.

5. (a) The Lagrange function is

$$L(p, \lambda) = \sum_{i=1}^{m} k_i \log p_i - \lambda \left(\sum_{i=1}^{m} p_i - 1 \right).$$

It is concave in p and is maximized for the $p(\lambda)$ such that

$$\frac{\partial L}{\partial p_i}(p, \lambda) = k_i \frac{1}{p_i} - \lambda = 0,$$

i.e., $p_i = k_i/\lambda$. The dual function is obtained by inserting this p in the Lagrange function

$$J(\lambda) = \sum_{i=1}^{m} k_i(\log k_i - \log \lambda) - \lambda \left(\sum_{i=1}^{m} k_i/\lambda - 1 \right)$$

$$J(\lambda) = \lambda - \log \lambda \sum_{i=1}^{m} k_i + \sum_{i=1}^{m} k_i \log k_i - \sum_{i=1}^{m} k_i = 0.$$

(b) Then

$$\frac{\partial J}{\partial \lambda}(\lambda) = 1 - \frac{\sum_{i=1}^{m} k_i}{\lambda} = 0.$$

Thus

$$\lambda = \sum_{i=1}^{m} k_i.$$

(c) With λ from (b) inserted in the expression for p in (a) we obtain:

$$p_i = \frac{k_i}{\sum_{i=1}^{m} k_i}.$$

(d) Note that $g(p) = 0$, λ is positive, so the point (p, λ) satisfies the GOC.