KTH Mathematics

Solutions for the exam in SF1811/SF1831/SF1841 Optimization for F.
Saturday March 13 2009 kl. 08.00–13.00

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There may be alternative solutions to the problem.

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1. (a) Setting the node potential $y_4 = 0$, and using that $c_{ij} = y_i - y_j$ for the ij corresponding to basic variables, then the rest of the node potentials are given by

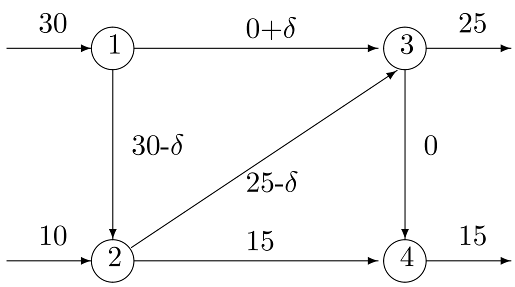
$$y_1 = 3, y_2 = 2, y_3 = 0.$$

Then the reduced costs for the other indices are given by $\hat{c}_{ij} = c_{ij} - y_i + y_j$:

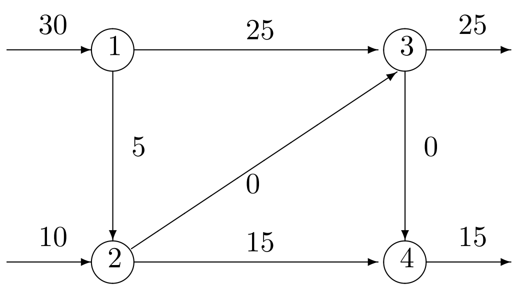
$$\hat{c}_{13} = 2 - 3 + 0 = -1, \hat{c}_{34} = 2 - 0 + 0 = 2,$$

Since the reduced cost $\hat{c}_{34}$ is negative, and the current BFS is not degenerate, the proposed solution is not optimal.

(b) We add the variable x_{13} to the basis. A cycle is created in the graph, and the maximal flow in this cycle is determined.



It is obtained for $\delta = 25$ and then the variable x_{23} becomes zero and exits the basis.



Again, setting the node potential $y_4 = 0$, and using that $c_{ij} = y_i - y_j$ for the ij corresponding to basic variables, then the rest of the node potentials are given by

$$y_1 = 3, y_2 = 2, y_3 = 1.$$

Then the reduced costs for the other indices are given by $\hat{c}_{ij} = c_{ij} - y_i + y_j$:

$$\hat{c}_{23} = 2 - 2 + 1 = 1, \hat{c}_{34} = 2 - 1 + 0 = 1,$$

Since the reduced costs are all positive the current solution is optimal.

(c) A spanning tree is a tree that connects all nodes, so that there is a path from any one node to any other node (not considering the directions of the arcs) and that has no cycles (which are paths that starts and ends in the same node without using the same arc twice)

There are 8 different spanning trees in the network graph.

2. (a) Alternative 1: Write the problem on standard form and verify with the simplex algorithm that it is optimal.

Alternative 2: First show that $x^{(a)} = (2,1,0)$ is feasible. Both constraints are satisfied, the inequality is satisfied with equality.

Consider the dual

$$(D) \quad \left[\begin{array}{ll} \max\limits_{y} & y_1 + 7y_2 \\ \text{s.t.} & y_1 + 2y_2 \leq 1 \\ & -y_1 + 3y_2 \leq -1 \\ & 2y_1 + y_3 \leq 3 \\ & y_2 \geq 0 \end{array} \right].$$

Since $x_1^{(a)}$ and $x_2^{(a)}$ are both non-zero, it follows by complementarity that $y_1 + 2y_2 = 1$ and $-y_1 + 3y_2 = -1$, i.e. $y_1 = 1$ and $y_2 = 0$. It is dual feasible and the optimal values of the dual and primal are the same; $y_1 + 7y_2 = 1 = x_1 - x_2 + 3x_3$.

(b) The problem (P) can be written on standard form

$$(P_s) \quad \left[\begin{array}{ll} \min\limits_{x} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \right]$$

where

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, \quad c = \begin{bmatrix} 6 & -1 & 3 & 0 \end{bmatrix}^T.$$

We start with x_1 and x_2 in the basis, giving the solution $x = (2,1,0,0)$ from (a).

$$B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, N = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

Then the equations $B^T y = c_B$ and $\hat{c}_N^T = c_N^T - y^T N$ gives

$$y = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \hat{c}_N^T = \begin{bmatrix} -6 & 1 \end{bmatrix}.$$

Let x_3 enter the basis. Which one should exit ?

From $B\hat{a}_3 = a_3$, we get $\hat{a}_3 = (7/5, -3/5)^T$, and since the second element is negative, x_1 exits the basis.

Update the basis and nonbasis matrices:

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, N = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

Then the equations $B^T y = c_B$ and $\hat{c}_N^T = c_N^T - y^T N$ gives

$$y = \frac{1}{7}\begin{bmatrix} 10 \\ 1 \end{bmatrix}, \quad \hat{c}_N^T = \begin{bmatrix} 30/7 & 1/7 \end{bmatrix}.$$

Since all reduced costs are nonnegative, the current bfs $\hat{x} = (0, \frac{13}{7}, \frac{10}{7}, 0)$ is optimal.

(c) The dual linear programing problem is

$$(D') \quad \begin{bmatrix} \max\limits_y & b^T y \\ \text{s.t.} & A^T y \le c \\ & y \text{ free} \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} \max\limits_y & y_1 + 7y_2 \\ \text{s.t.} & y_1 + 2y_2 \le 6 \\ & -y_1 + 3y_2 \le -1 \\ & 2y_1 + y_2 \le 3 \\ & -y_2 \le 0 \end{bmatrix}.$$

Since $x_2^{(a)}$ and $x_3^{(a)}$ are both non-zero, it follows by complementarity that $-y_1 + 3y_2 = -1$ and $2y_1 + y_2 = 3$, i.e. $y_1 = 10/7$ and $y_2 = 1/7$. This solution also satisfies constraints 1 and 4, so it is feasible for the dual.

The dual solution can also be obtained from the last simplex iteration in (b).

3. (a) The Hessian of $e^{x_1^2+x_2^2+x_3^2}$, at the origin is

$$H = 2\begin{bmatrix} 1+2x_1^2 & 2x_1x_2 & 2x_1x_3 \\ 2x_1x_2 & 1+2x_2^2 & 2x_2x_3 \\ 2x_1x_3 & 2x_2x_3 & 1+2x_3^2 \end{bmatrix} e^{x_1^2+x_2^2+x_3^2}\Bigg|_{x=0} = 2I.$$

Therefore,

$$\nabla^2 f(0) = \begin{bmatrix} 2 & -2 & 4 \\ -2 & 5 & 2 \\ 4 & 2 & 16 \end{bmatrix}.$$

The LDL^T factorization is given by

$$\nabla^2 f(0) = \begin{bmatrix} 2 & -2 & 4 \\ -2 & 5 & 2 \\ 4 & 2 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $D_{33} = -4$, the Hessian is not positive definite, since the eigenvalues of the Hessian have the same signs as the diagonal elements of D. (One eigenvalue of the Hessian will be negative and two of them will be positive)

(b) All the diagonal elements in the given LDL^T factorization are non-negative, so the modified Hessian is positive semidefinite. But since one diagonal element is zero, the modified Hessian is not positive definite.

(c) $-c$ is the same as minus the first column in $\tilde{H}$, so $-c = \tilde{H}(-e_1)$, where $e_1 = (1,0,0)^T$.

(d) Applying row operations to $\tilde{H}$ you get:

$$\tilde{H} = \nabla^2 f(0) = \begin{bmatrix} 2 & -2 & 4 \\ -2 & 5 & 2 \\ 4 & 2 & 20 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 4 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The null space of $\tilde{H}$ is spanned by

$$Z = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix},$$

and this is clearly a one-dimensional space.

4. (a) Clearly, $\nabla f(x)^T = Hx + c$, and then $\nabla f(x^{(0)})^T = Hx^{(0)} + c = c$. Then,

$$(d^{(1)})^T c = -1, \quad (d^{(2)})^T c = -3, \quad (d^{(3)})^T c = 2$$

so the first two are descent directions, but not the third one. The first and third ones are feasible directions, since if all components are positive, the vectors points in to the feasible region. The second one, on the other hand, points outside the feasible region and is not a feasible direction. Thus, only $d^{(1)}$ is a feasible descent direction.

Since there exists a feasible descent direction at $x^{(0)}$ it can not be a local minimum.

(b) Let

$$\begin{aligned} g_1(x) &= 1 - (x_1^2 + x_2^2) \\ g_2(x) &= x_1 \\ g_3(x) &= x_2 \\ g_4(x) &= x_3 \end{aligned}$$

Then, the problem can be written min $f(x)$ subject to $g_i(x) \geq 0$ for $i = 1,2,3,4$. At $x^{(1)}$, constraints 1,3 and 4 are active.

$$\nabla f(x^{(1)})^T = Hx^{(1)} + c = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad \nabla g_1(x^{(1)})^T = \begin{pmatrix} -2x_1^{(1)} \\ -2x_2^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

$$\nabla g_3(x)^T = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \nabla g_4(x)^T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

For the KKT-conditions to be satisfied we need to find non-negative Lagrange parameters such that

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \lambda_1 - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \lambda_3 - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \lambda_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now $\lambda_1 = 1/2, \lambda_3 = 2, \lambda_4 = 1$ and $\lambda_2 = 0$.
So the KKT-conditions are satisfied.

(c) The point $\bar{x} = (0, 0, 0)$ is feasible. A nullspace matrix Z for A is given by

$$Z = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and } Z^T H Z = 12, \quad c^T Z = 1$$

so $v = -1/12$ solves $Z^T H Z v = -Z^T c$. Then, since the reduced Hessian $Z^T H Z$ is positive definite, $\hat{x} = \bar{x} + vZ = (-1/12, -1/12, 0)$ is the minimum.

(d) Note that the reduced Hessian is positive definite, so the $f(x)$ is strictly convex on the feasible set. Therefore, the stationary point determined in (c) is a global minimum.

5. (a) The Lagrange function is

$$L(x, \lambda) = \sum_{i=1}^{m} \frac{a_i}{x_i} - \lambda \left(b - \sum_{i=1}^{m} \log x_i \right) = -\lambda b + \sum_{i=1}^{m} \left(\frac{a_i}{x_i} + \lambda \log x_i \right)$$

It is well defined for $x_i > 0$, separates into m independent minimization problems, $\min_{x_i > 0} \ell_i(x_i, \lambda)$ where $\ell_i(x_i, \lambda) = a_i/x_i + \lambda \log x_i$, for each x_i, and is minimized for the $x_i(\lambda)$ such that

$$\frac{\partial L}{\partial x_i}(x, \lambda) = -\frac{a_i}{x_i^2} + \frac{\lambda}{x_i} = 0,$$

i.e., $x_i(\lambda) = a_i/\lambda$. This follows since it is the only point where the derivative is zero and the second derivative $\ell_i''(a_i/\lambda, \lambda) = \lambda/x_i^2 > 0$. (Note that ℓ_i goes to infinity as x_i goes to zero or infinity)
The dual function is obtained by inserting this x_i in the Lagrange function

$$J(\lambda) = \sum_{i=1}^{m} \frac{a_i}{a_i/\lambda} - \lambda \left(b - \sum_{i=1}^{m} \log(a_i/\lambda) \right)$$

$$J(\lambda) = m\lambda - m\lambda \log \lambda + \lambda \sum_{i=1}^{m} \log a_i - \lambda b$$

(b) We know that the dual function J is concave and it is differentiable, so the maximum point must satisfy

$$\frac{\partial J}{\partial \lambda}(\lambda) = m - m - m \log \lambda + \sum_{i=1}^{m} \log a_i - b = 0$$

Thus
$$\lambda = \exp\left\{ (\sum_{i=1}^{m} \log a_i - b)/m \right\}.$$

(c) With λ from (b) inserted in the expression for x_i in (a) we obtain:
$$\hat{x}_i = \frac{a_i}{\lambda} = a_i \exp\left\{ \frac{b}{m} - \sum_{i=1}^{m} \frac{\log a_i}{m} \right\}.$$

(d) Note that $g(\hat{x}) = 0$, λ is positive, so the point $(\hat{x}, \lambda)$ satisfies the GOC.