KTH Mathematics

Solutions for the exam in SF1811/41 Optimization.
Monday June 11, 2012, tid. 14.00–19.00

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There may be alternative solutions.

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1. (a) There are three equations that should hold if the measurements were exact.

$$U_1 = I_1 R_1, \quad \text{d.v.s } 12 = 0.3 R_1$$

$$U_2 = I_2 R_2, \quad \text{d.v.s } 12 = 0.4 R_2$$

$$U_3 = I_3 R_3, \quad \text{d.v.s } 12 = 0.6/(1/R_1 + 1/R_2)$$

And with the conductivity as the unknown variables

$$12L_1 = 0.3, \quad 12L_2 = 0.4, \quad 12(L_1 + L_2) = 0.6.$$

The lest-squares problem is to minimize the sum of squares of the errors in these equations, i.e., to minimize

$$f(L_1, L_2) = (12L_1 - 0.3)^2 + (12L_2 - 0.4)^2 + (12(L_1 + L_2) - 0.6)^2.$$

This can be written as $f(L) = \|AL - b\|^2$ where $L = [L_1,\ L_2]^T$ and

$$A = \begin{bmatrix} 12 & 0 \\ 0 & 12 \\ 12 & 12 \end{bmatrix}, \quad b = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.6 \end{bmatrix}.$$

The optimal solution is then given by $L = (A^T A)^{-1} A^T b = (1/45, 11/360)$ and the corresponding resistances are $R = (45, 360/11)$.

(b) We start by transforming A in stair-case form by elementary row-operations:

$$A = \begin{bmatrix} 2 & 4 & 1 & 2 \\ 4 & 4 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/2 & 1 \\ 0 & -4 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & -1/2 \\ 0 & 1 & 1/2 & 3/4 \end{bmatrix}$$

Since the two first columns are unit vectors, the range space is spanned by the first two columns in A, i.e., the range space is spanned by the vectors $[2,\ 4]^T$ and $[4,\ 4]^T$.

The nullspace is spanned by the vectors

$$\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1/2 \\ -3/4 \\ 0 \\ 1 \end{bmatrix}.$$

2. (a) The flow at each node is in balance if $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ -3 \\ -2 \\ -5 \end{bmatrix},$$

(b) The basic solution is found by considering the flow through the spanning tree and make sure that the flow is in balance at each node, the basic variables are then given by

$$x_{12} = 0, \quad x_{13} = 8, \quad x_{14} = 2, \quad x_{35} = 8, \quad x_{46} = 2, \quad x_{57} = 5.$$

It is degenerate since $x_{12} = 0$, even though it is a basic variable.

(c) Låt nodpotentialen Let the node potential y_7 at node 7 be 0. $y_5 - y_7 = c_{57} = 3$ gives $y_5 = 3$.
$y_3 - y_5 = c_{35} = 2$ gives $y_3 = 5$.
$y_1 - y_3 = c_{13} = 3$ gives $y_1 = 8$.
$y_1 - y_2 = c_{12} = 2$ gives $y_2 = 6$.
$y_1 - y_4 = c_{14} = 1$ gives $y_4 = 7$.
$y_4 - y_6 = c_{46} = 1$ gives $y_6 = 6$.
The reduced costs are now $r_{25} = c_{25} - y_2 + y_5 = 4-6+3 = 1$, $r_{36} = c_{36} - y_3 + y_6 = 3 - 5 + 6 = 4$. and $r_{67} = c_{67} - y_6 + y_7 = 2 - 6 + 0 = -4$. Since the reduced cost r_{67} is negative, x_{67} it will enter the basis. We add the corresponding arc to the sub-graph and then a cycle is formed. The variable x_{67} can be increased to 5 whereby x_{57} becomes zero and exits the basis. A new basic solution is then given by

$$x_{12} = 0, \quad x_{13} = 3, \quad x_{14} = 7, \quad x_{35} = 3, \quad x_{46} = 7, \quad x_{67} = 5.$$

(also degenerate)
Låt nodpotentialen y_7 vid nod 7 vara 0. Let the node potential y_7 at node 7 be 0. $y_6 - y_7 = c_{67} = 2$ gives $y_6 = 2$.
$y_4 - y_6 = c_{46} = 1$ gives $y_4 = 3$.
$y_1 - y_4 = c_{14} = 1$ gives $y_1 = 4$.
$y_1 - y_2 = c_{12} = 2$ gives $y_2 = 2$.
$y_1 - y_3 = c_{13} = 3$ gives $y_3 = 1$.
$y_3 - y_5 = c_{35} = 2$ gives $y_5 = -1$.
The reduced costs are now $r_{25} = c_{25} - y_2 + y_5 = 4-2+1 = 1$, $r_{36} = c_{36} - y_3 + y_6 = 3 - 1 + 2 = 4$. and $r_{57} = c_{57} - y_5 + y_7 = 3 - (-1) + 0 = 4$. Since all reduced costs are positive, the current basic solution is the unique optimum.

3. (a) We start with x_1 and x_2 as basic variables. I.e. basic and non-basic variable indices are $\beta = \{1, 2\}$ and $\eta = \{3, 4\}$, so

$$B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and $\bar{b} = B^{-1}b = [1\ 1]^T$ which gives the starting basic solution $x = (1, 1, 0, 0)$.

From the equations $B^T y = c_B$ and $\hat{c}_N tT = c_N^T - y^T N$ we get

$$y = \begin{bmatrix} -2 \\ 5/2 \end{bmatrix}, \quad r_N^T = \begin{bmatrix} -3/2 & 1/2 \end{bmatrix}.$$

Let x_3 enter the basis. Which one should exit ?

From $B\hat{a}_4 = a_4$, we get that $\hat{a}_4 = (3/2, -1/2)^T$, and since the first element is the only positive one, x_1 exits the basis.

Update basic and non-basic matrices; The basic and non-basic variable indices are given by $\beta = \{2, 3\}$ och $\eta = \{1, 4\}$, and

$$B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, N = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The equations $B^T y = c_B$ and $\hat{c}_N^T = c_N^T - y^T N$ gives

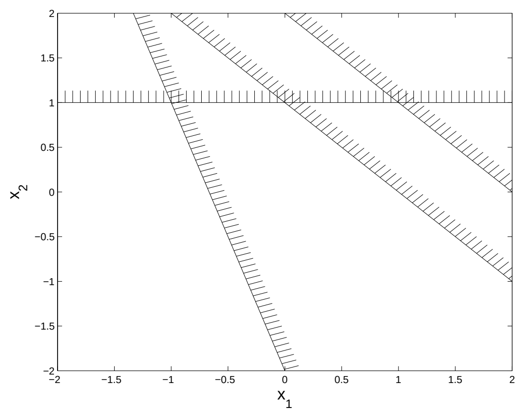
$$y = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad r_N^T = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

Since all reduced costs are non-negative, $\hat{x} = (0, 4/3, 2/3, 0)^T$ is optimal.

(b) The dual is

$$(D) \quad \begin{bmatrix} \min\limits_{y} & 4y_1 + 2y_2 \\ \text{s.t.} & y_1 + y_2 \leq 2 \\ & 3y_1 + y_2 \leq -2 \\ & y_2 \leq 1 \\ & y_1 + y_2 \leq 1 \end{bmatrix}.$$

(c) The feasible region for the dual is given by

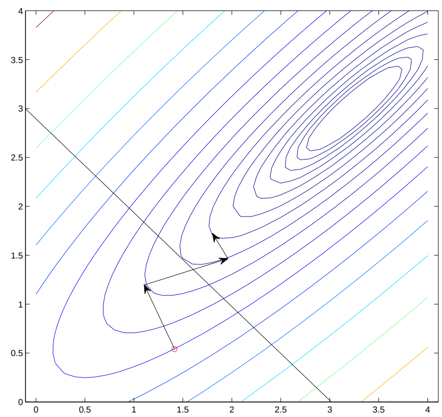


The complementarity condition tells us that $\hat{x}^T(A^T\hat{y}-c) = (0, 4/3, 2/3, 0)^T(A^T y - c) = 0$, hence $3y_1 + y_2 = -2$ and $y_2 = 1$ must hold so the optimum for the dual is obtained at $y = (-1, 1)$.

Since we have equality constraints in the primal it always hold that $\hat{y}^T(A\hat{x} - b)$ since $\hat{x}$ is feasible for the primal.

It is easy to also check that $b^T y = -4 + 2 = -2$, $c^T x = 4/3 * (-2) + 2/3 = -6/3 = -2$, are both the same.

4. (a) The iterations for the gradient method can be seen in the Figure below.



What is important to notice is that the search directions are orthogonal to the level curve that passes through the starting point of the iteration, and that the step length is chosen so that search direction coincides with the tangent of a level curve in the next point of the iteration. This leads to that the next search direction will become orthogonal to the previous one. The step of Newton's method will point directly to the global minimum since the objective function is a convex quadratic function.

(b) The matrix H can be factorized as $H = LDL^T$ where

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Since the diagonal elements in D are positive, the matrix H is positive definite and the quadratic objective function is convex.

(c) The solution to the problem is unique since H is positive definite, and it is given by the x which solves $Hx = -c$, i.e., by $x = (13/4,\ 3)$.

(d) First we determine the nullspace of A, it is spanned by

$$Z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

A solution $\bar{x}$ to the constraint is given by

$$\bar{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

The equation system $(ZHZ^T)v = -Z^T(H\bar{x}+c)$, i.e. $17v = -(-26)$, gives $v = 26/17$. Finally, $\hat{x} = \bar{x} + Zv = [26/17\ 25/17]^T$ is a global minimum since the problem is convex.

5. (a) The objective function $f(x)$ is the sum of the convex functions $e^{x_1+x_2}$ and $-x_1 - 3x_2$, so it is convex.

The feasible region is the intersection of two sets. Since the intersection is not a connected set, but consists of two separate sets, it is not convex.

Since the feasible region is not convex, the optimization problem is not convex.

(b) If we know that the sets $\mathcal{C}_1$ and $\mathcal{C}_2$ both are convex, then we know that their union is also convex.

If the sets $\mathcal{C}_1$ and $\mathcal{C}_2$ are not convex, but their intersection is only one point, then the intersection is convex. But if instead $\mathcal{C}_1 = \mathcal{C}_2$, then the intersection is $\mathcal{C}_1$ which is by assumption not convex. So we can not say anything about the convexity in this case.

(c) Let

$$g_1(x) = -e^{x_1} + x_2, \quad g_2(x) = e \cdot x_1^2 - x_2.$$

Then

$$\nabla f(x) = \begin{bmatrix} e^{x_1+x_2} - 1 & e^{x_1+x_2} - 3 \end{bmatrix},$$
$$\nabla g_1(x) = \begin{bmatrix} -e^{x_1} & 1 \end{bmatrix}, \quad \nabla g_2(x) = \begin{bmatrix} 2e \cdot x_1 & -1 \end{bmatrix}.$$

In $\mathbf{x}^{(c)} = (0,0)$ it holds that $g_1(\mathbf{x}^{(c)}) = -1$ and $g_2(\mathbf{x}^{(c)}) = 0$. Constraint 1 is not active and therefore must $\hat{y}_1 = 0$.

But then the equation

$$\nabla f(\mathbf{x}^{(c)}) + \hat{y}_1 \nabla g_1(\mathbf{x}^{(c)}) = \begin{bmatrix} e^0 - 1 & e^0 - 3 \end{bmatrix} + \hat{y}_2 \begin{bmatrix} 0 & -1 \end{bmatrix} = 0$$

has no solution and the KKT-conditions are not satsified. Therefore, $\mathbf{x}^{(c)} = (0,0)$ can not be a local minimum.

(d) In $\mathbf{x}^{(d)} = (1,e)$ it holds that $g_1(\mathbf{x}^{(c)}) = 0$ and $g_2(\mathbf{x}^{(c)}) = 0$. Both constraints are thus active and both $\hat{y}_1$ and $\hat{y}_2$ can be non-zero.

Then the equation

$$\nabla f(\mathbf{x}^{(d)}) + \hat{y}_1 \nabla g_1(\mathbf{x}^{(d)}) + \hat{y}_2 \nabla g_2(\mathbf{x}^{(d)}) =$$
$$= \begin{bmatrix} e^{1+e} - 1 & e^{1+e} - 3 \end{bmatrix} + \hat{y}_1 \begin{bmatrix} -e^1 & 1 \end{bmatrix} + \hat{y}_2 \begin{bmatrix} 2e^1 & -1 \end{bmatrix} = 0$$

has a solution

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} -e & 2e \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} e^{1+e} - 1 \\ e^{1+e} - 3 \end{bmatrix} = \frac{1}{3e} \begin{bmatrix} 1 & 2e \\ 1 & e \end{bmatrix} \begin{bmatrix} e^{1+e} - 1 \\ e^{1+e} - 3 \end{bmatrix}$$

which is positive and the KKT-conditions are satisfied.

Therefore, $\mathbf{x}^{(c)} = (0,0)$ can be a local minimum, but since the problem is not convex the KKT-conditions are only necessary, but not sufficient, so we can not say that it is a local minimum.

(e) In an interior point no constraints are active and then both $\hat{y}_1$ and $\hat{y}_2$ must be equal to zero. Since $\nabla f(x) = \begin{bmatrix} e^{x_1+x_2} - 1 & e^{x_1+x_2} - 3 \end{bmatrix}$ there are no such point such that the derivative is zero, hence the KKT-conditions can not be satisfied in an interior point and there are no local minima of the function in an interior point.