KTH Mathematics

Solutions for the exam in Optimization.
wednesday March 13, 2013, time. 8.00–13.00

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There may be alternative solutions to the problem.

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1. (a) The given flow is $x = (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = (6, 0, 0, 4, 0, 1)$, so x_{12},x_{23} and x_{24} are the basic variables corresponding to a spanning tree in the graph.

Put the node potential y_4 at node 4 to be 0.
Then $y_3 - y_4 = c_{34} = 1$ gives $y_3 = 1$.
Then $y_2 - y_3 = c_{23} = 3$ gives $y_2 = 4$.
Then $y_1 - y_2 = c_{12} = 5$ gives $y_1 = 9$.

The reduced costs are now $r_{13} = c_{13} - y_1 + y_3 = -4$, $r_{14} = c_{14} - y_1 + y_4 = -6$ and $r_{24} = c_{24} - y_2 + y_4 = -1$. Since the reduced cost r_{14} is most negative the flow in x_{14} should be increased. Increasing the flow in x_{14} to t a cycle in the graph is created and we must compensate to get $x_{34} = 1 - t$, $x_{23} = 4 - t$ and $x_{12} = 6 - t$. So t can become at most 1 and then x_{34} becomes zero and exits the basis.

In the new flow $x = (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = (5, 0, 1, 3, 0, 0)$, so x_{12}, x_{14} and x_{23} are the basic variables corresponding to a spanning tree in the graph.

Put the node potential y_4 at node 4 to be 0.
Then $y_1 - y_4 = c_{14} = 3$ gives $y_1 = 3$.
Then $y_1 - y_2 = c_{12} = 5$ gives $y_2 = -2$.
Then $y_2 - y_3 = c_{23} = 3$ gives $y_3 = -5$.

The reduced costs are now $r_{13} = c_{13} - y_1 + y_3 = -4$, $r_{24} = c_{14} - y_1 + y_4 = 5$ and $r_{34} = c_{24} - y_2 + y_4 = 6$. Since the reduced cost r_{13} is negative the flow in x_{13} should be increased. Increasing the flow in x_{13} to t a cycle in the graph is created and we must compensate to get $x_{23} = 3 - t$, $x_{12} = 5 - t$ and $x_{13} = t$. So t can become at most 3 and then x_{23} becomes zero and exits the basis.

In the new flow $x = (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = (2, 3, 1, 0, 0, 0)$, so x_{12}, x_{13} and x_{14} are the basic variables corresponding to a spanning tree in the graph.

Put the node potential y_4 at node 4 to be 0.
Then $y_1 - y_4 = c_{14} = 3$ gives $y_1 = 3$.
Then $y_1 - y_2 = c_{12} = 5$ gives $y_2 = -2$.
Then $y_1 - y_3 = c_{13} = 4$ gives $y_3 = -1$.

The reduced costs are now $r_{23} = c_{23} - y_2 + y_3 = 4$, $r_{24} = c_{24} - y_2 + y_4 = 4$ and $r_{34} = c_{34} - y_3 + y_4 = 2$.

Since the reduced costs are positive the flow is optimal.

(b) If A is the adjacency matrix, then $Ax = b$ and $x \geq 0$ are the contraint.

Let d be a vector in the nullspace of A and assum that x is the current flow.

For any value of ϵ it holds then that $A(x + \epsilon d) = Ax + \epsilon Ad = Ax = b$, so the first constraint is satisfied for any choice of ϵ. The second constraint $x + \epsilon d \geq 0$

will be satisfied for any small enough positive value of ϵ if the elements in d that corresponds to elements in x that are zero are non-negative.

Furthermore, d is a feasible descent direction if $c^T d < 0$, where c is the cost vector.

The vector $d = (d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}) = (-1, 0, +1, -1, 0, -1)$ defines a loop in the network that wass used to change from the first basis to the second one.

2. (a) The standard form is

$$(P_s) \quad \left[\begin{array}{ll} \min\limits_{x} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}\right]$$

Define slack variables x_4 and x_5, then $c = -(2\ 3\ 3\ 0\ 0)^T$. The constraints are defined by

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

We start with x_4 and x_5 as basic variables. I.e. basic and non-basic variable indices are $\beta = \{1, 2, 3\}$ and $\eta = \{4, 5\}$, so

$$A_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_\nu = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

and $\bar{b} = A_\beta^{-1} b = [3\ 3]^T$ which gives the starting basic solution $x = (0, 0, 0, 3, 3)$. From the equations $A_\beta^T y = c_\beta$ and $\hat{c}_\nu^T = c_\nu^T - y^T A_\nu$ we get

$$y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad r_\nu^T = \begin{bmatrix} -2 & -3 & -3 \end{bmatrix}.$$

Let x_2 enter the basis. Which one should exit ?

From $A_\beta \hat{a}_4 = a_4$, we get that $\hat{a}_4 = (1, 2)^T$, and since x_5 becomes zero first, "smallest quotient", x_5 exits the basis.

Update basic and non-basic matrices; The basic and non-basic variable indices are given by $\beta = \{2, 4\}$ och $\eta = \{1, 3, 5\}$, and

$$A_\beta = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, A_\nu = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The equations $A_\beta^T y = c_\beta$ and $\hat{c}_\nu^T = c_\nu^T - y^T A_\nu$ gives

$$y = \begin{bmatrix} 0 \\ -1.5 \end{bmatrix}, \quad r_\nu^T = \begin{bmatrix} -0.5 & -1.5 & 1.5 \end{bmatrix}.$$

Let x_3 enter the basis. Which one should exit ?

From $A_\beta \hat{a}_3 = a_3$, we get that $\hat{a}_3 = (0.5, 1.5)^T$, $\bar{b} = A_\beta^{-1} b = [1.5\ 1.5]^T$ and since x_4 becomes zero first, "smallest quotient", x_4 exits the basis.

Update basic and non-basic matrices; The basic and non-basic variable indices are given by $\beta = \{2, 3\}$ och $\eta = \{1, 4, 5\}$, and

$$A_\beta = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, A_\nu = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The equations $A_\beta^T y = c_\beta$ and $\hat{c}_\nu^T = c_\nu^T - y^T A_\nu$ gives

$$y = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad r_\nu^T = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}.$$

Since all reduced costs are non-negative, $\hat{x} = (0, 1, 1, 0, 0)^T$ is optimal.

(b) The dual is

$$(D) \quad \begin{bmatrix} \min\limits_{y} & 3y_1 + 3y_2 \\ \text{s.t.} & y_1 + y_2 \geq 2 \\ & y_1 + 2y_2 \geq 3 \\ & 2y_1 + y_2 \geq 3 \\ & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0. \end{bmatrix}.$$

(c) A graphical description of the problem shows that there is not a unique solution, all points on the line $y_1 + y_2 = 2$ that are feasible are also optimal. $y_1 = 1, y_2 = 1$ is one optimal solution.
All y are non-negative.
$y_1 + y_2 = 2 \geq 2$
$y_1 + 2y_2 = 3 \geq 3$
$2y_1 + y_2 = 3 \geq 3$

The primal problem has the optimal solution $\hat{x}_1 = 0, \hat{x}_2 = 1, \hat{x}_3 = 1$.
$x_1 + x_2 + 2x_3 = 3 \leq 3$,
$x_1 + 2x_2 + 1x_3 = 3 \leq 3$.
All the x are non-negative.
Finally, the complementarity conditions
$y_1(x_1 + x_2 + 2x_3 - 3) = 1 \cdot 0 = 0$,
$y_2(x_1 + 2x_2 + 1x_3 - 3) = 1 \cdot 0 = 0$.
$x_1(y_1 + y_2 - 2) = 0 \cdot 0 = 0$
$x_2(y_1 + 2y_2 - 3) = 1 \cdot 0 = 0$
$x_3(2y_1 + y_2 - 3) = 1 \cdot 0 = 0$

are all satisfied, so the x and y are optimal to the respective problems.

3. (a) For f to be convex on the whole $\mathbb{R}^3$ it is necessary that the matrix H is positive semidefinite. Use LDL^T-factorization:

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the diagonal elements in the D-matrix are positive, the matrix H is positive definite and the function is convex on $\mathbb{R}^3$.

(b) Use Gauss-Jordan to determine a basis for the nullspace of A. With the nullspace method a Z-matrix and $\bar{x}$ given by

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

the equation system $(ZHZ^T)v = -Z^T(H\bar{x} + c)$:

$$\begin{bmatrix} 9 & 5 \\ 5 & 4 \end{bmatrix} v = \begin{bmatrix} 4 \\ 1 \end{bmatrix},$$

yields $v = [1, \ -1]^T$. Therefore, $\hat{x} = \bar{x} + Zv = [1 \ 0 \ 0]^T$ is a global minimum since the problem is convex.

(c) For the Lagrange method, the following equation system must be solved

$$\begin{bmatrix} H & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix},$$

That is:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 6 & 6 & 1 \\ 1 & 6 & 10 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

from which we see that $\hat{u} = -2$ and $\hat{x}$ as above.

4. (a) The gradient and hessian are given by

$$\nabla f(x,y) = \begin{bmatrix} 1 + \log(\frac{x}{y}) & -\frac{x}{y} + 1/e \end{bmatrix}, \quad \nabla^2 f(x) = \begin{bmatrix} \frac{1}{x} & -\frac{1}{y} \\ -\frac{1}{y} & \frac{x}{y^2} \end{bmatrix}.$$

The first order optimality conditions are satisfied when $\nabla f(x,y) = 0$, i.e., when $x/y = 1/e$ and $x \geq 0.1$ and $y \geq 0.1$.

The second order optimality conditions depends on the definiteness of the Hessian, here multiplying the first row with x/y and adding to the second we get

$$\begin{bmatrix} \frac{1}{x} & -\frac{1}{y} \\ -\frac{1}{y} + \frac{1}{x}x/y & \frac{x}{y^2} - \frac{1}{y}x/y \end{bmatrix} = \begin{bmatrix} \frac{1}{x} & -\frac{1}{y} \\ 0 & 0 \end{bmatrix}$$

so the Hessian is positive semidefinite but not positive definite. So the second order necessary conditions are satisfied but not the neccesary conditions. So we can not say that the points are not optimal (minimal), and we can not say that they are optimal.

However, since the function is convex and the gradient is equal to zero we can say that all the feasible points on the line determined above are minimizing the function.

(b) We consider the Lagrange optimality conditions

$$\nabla f(x) + \lambda \nabla h(x) = \left[\begin{array}{cc} 1 + \log(\frac{x}{y}) & -\frac{x}{y} + 1/e \end{array}\right] + \lambda \left[\begin{array}{cc} 1 & 1 \end{array}\right] = 0,$$

which is satisfied for $\lambda = 0$ and $x/y = 1/e$. Using the constraint $x + y = 1$ gives $x = \frac{1}{1+e}$ and $y = \frac{e}{1+e}$.
(Since $\lambda = 0$ we know that the optimal objective value of the problem would not change if we change the constraint to $x + y = 1 + \delta$. The objective function is in fact zero for all values on the line $x/y = 1/e$.)

5. (a) Use the definition of a convex set. Assume that (t, x) and (s, y) both belong to $\operatorname{epi}(f_0)$. Then we want to show that for arbitrary $\lambda \in (0, 1)$

$$(r, z) = \lambda(t, x) + (1 - \lambda)(s, y) = (\lambda t + (1 - \lambda)s, \lambda x + (1 - \lambda)y)$$

belong to $\operatorname{epi}(f_0)$, i.e., that $f_0(z) \leq r$. Now

$$f_0(z) = f_0(\lambda x + (1 - \lambda)y) \leq \lambda f_0(x) + (1 - \lambda) f_0(y) \leq \lambda t + (1 - \lambda)s = r$$

follows from convexity of f_0 (first inequality) and that (t, x) and (s, y) both belong to $\operatorname{epi}(f_0)$ (second inequality).

(b) Assume that (t, x) satisfies the KKT conditions for (P_ℓ), i.e.

$$(1, 0) + y_0(-1, \nabla f_0) + \sum_{i=1}^{m} y_i(0, \nabla f_i(x)) = 0$$

$$f_0(x) - t \leq 0, \quad f_i(x) \leq 0, i = 1, \cdots, m.$$

$$y_0, y_1, \cdots, y_m \geq 0.$$

$$y_0(f_0(x) - t) = 0, \quad y_i f_i(x) = 0, i = 1, \cdots, m.$$

The first condition say that $y_0 = 1$ ($y_0 \geq 0$ ok) and

$$y_0 \nabla f_0 + \sum_{i=1}^{m} y_i \nabla f_i(x) = 0$$

which shows that KKT 1 for (P) is satisfied.
The last one say that $t = f_0(x)$, since $y_0 \neq 0$. What remains is

$$f_i(x) \leq 0, i = 1, \cdots, m.$$

$$y_1, \cdots, y_m \geq 0.$$

$$y_i f_i(x) = 0, i = 1, \cdots, m.$$

which shows that KKT 2-4 for (P) is satisfied.

(c) The gradients are given by

$$\nabla f_0(x) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \nabla f_1(x) = \begin{bmatrix} -1 & 2x_2 & 2x_3 \end{bmatrix}, \quad \nabla f_2(x) = \begin{bmatrix} 0 & -1 & -1 \end{bmatrix}.$$

The KKT conditions are

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + y_1 \begin{bmatrix} -1 & 2x_2 & 2x_3 \end{bmatrix} + y_2 \begin{bmatrix} 0 & -1 & -1 \end{bmatrix} = 0.$$

$$x_2^2 + x_3^2 - x_1 \leq 0, \quad 1 - x_2 - x_3 \leq 0.$$

$$y_1 \geq 0, \quad y_2 \geq 0.$$

$$y_1(x_2^2 + x_3^2 - x_1) = 0, \quad y_2(1 - x_2 - x_3) = 0.$$

We see from KKT 1 that $y_1 = 1$, and then from KKT 4 $x_2^2 + x_3^2 = x_1$.
From KKT 1 we also get

$$2x_2 = y_2 \quad 2x_3 = y_2$$

If $y_2 = 0$, then $x_2 = x_3 = 0$ and then also $x_1 = 0$. But then $1 - x_2 - x_3 = \not\leq 0$. So $y_2 > 0$, and then $1 - x_2 - x_3 = 1 - y_2 = 0$, so $y_2 = 1$ and $x_2 = x_3 = 1/2$ and then $x_1 = 1/4 + 1/4 = 1/2$. This point then satisfies all KKT conditions.
Since the optimization problem is convex the point $x_1 = x_2 = x_3 = 1/2$ is the global minimum.