(1) (a) The problem (P) is in canonical form:

$$(P): \begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{cases}$$

where

$$c = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 2010 \\ 2011 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & 0 & \\ 1 & 1 & 1 & & \\ \vdots & & & \ddots & \\ 1 & \cdots & \cdots & & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 2010 \\ 2011 \end{bmatrix}$$

We have that

$$\hat{x} := 2011\, e_1 = \begin{bmatrix} 2011 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \geq 0$$

and

$$A\hat{x} = \begin{bmatrix} 2011 \\ 2011 \\ \vdots \\ 2011 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 2011 \end{bmatrix} = b,$$

and so $\hat{x}$ is feasible for (P).

The dual problem (D) to (P) is given by

$$(D): \begin{cases} \text{maximize} & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geq 0 \end{cases}$$

that is,

$$\begin{cases} \text{maximize} & y_1 + 2y_2 + 3y_3 + \dots + 2011\,y_{2011} \\ \text{s.t.} & y_1 + y_2 + \dots + y_{2011} \leqslant 1 \\ & y_2 + \dots + y_{2011} \leqslant 2 \\ & y_3 + \dots + y_{2011} \leqslant 3 \\ & \vdots \\ & y_{2010} + y_{2011} \leqslant 2010 \\ & y_{2011} \leqslant 2011, \\ & y_1 \geqslant 0,\ y_2 \geqslant 0,\ y_3 \geqslant 0,\ \dots,\ y_{2011} \geqslant 0. \end{cases}$$

We have that

$$\hat{y} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \geqslant \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and $A^T\hat{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \leqslant \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 2011 \end{bmatrix} = c,$

and so $\hat{y}$ is feasible for (D).

We have $c^T\hat{x} = 2011 + 2\cdot 0 + 3\cdot 0 + \dots + 2011\cdot 0$

$= 2011,$

and $b^T\hat{y} = 0 + 2\cdot 0 + 3\cdot 0 + \dots + 2011\cdot 1$

$= 2011.$

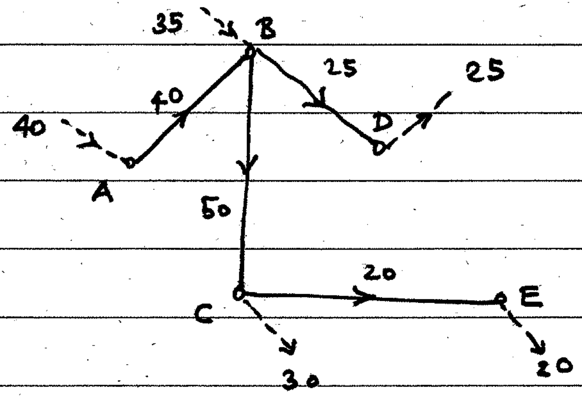
Since (1) $\hat{x}$ is feasible for (P),
(2) $\hat{y}$ is feasible for (D), and
(3) $c^T\hat{x} = b^T\hat{y}$,

it follows that $\hat{x}$ is optimal for (P).
(By weak duality, for any feasible x for (P), $c^Tx \geqslant b^T\hat{y} = c^T\hat{x}$.)

(1)(b) The network is balanced, since

$$40 + 35 = 75 = 30 + 25 + 20.$$

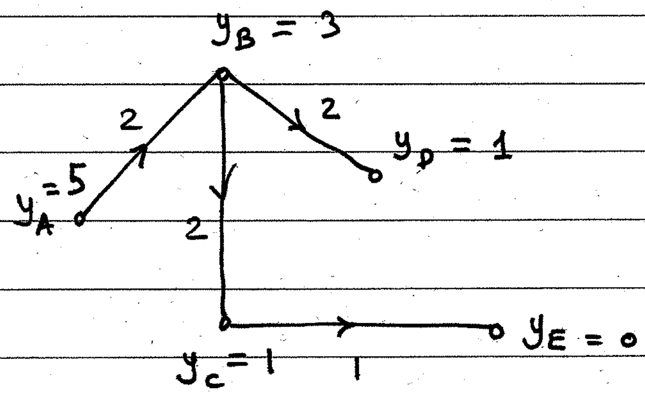
The basic solution can be obtained by using flow balance:



As all flows are ≥ 0, it is a basic feasible solution.

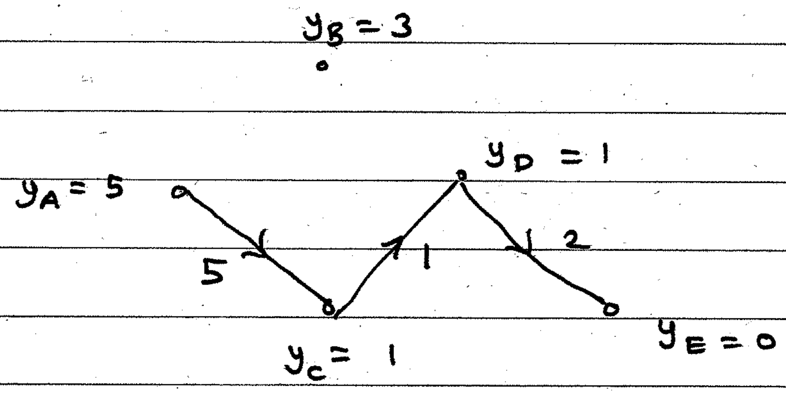
The simplex multipliers vector y can be determined using

$$\left.\begin{array}{l} c_{ij} = y_i - y_j \\ y_m = 0 \end{array}\right\} \text{ for tree edges } (i,j):$$



The reduced costs of the nonbasic variables can be found using

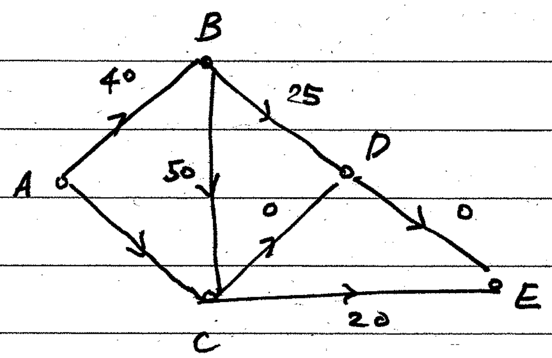
$$r_{ij} = c_{ij} - (y_i - y_j) \quad \text{for the non-tree edges } (i,j).$$



$$r_{AC} = 5 - (5-1) = 5 - 4 = 1 \geq 0,$$
$$r_{CD} = 1 - (1-1) = 1 - 0 = 1 \geq 0$$
$$r_{DE} = 2 - (1-0) = 2 - 1 = 1 \geq 0.$$

As $r_{ij} \geq 0$, the current basic feasible solution is optimal.

The optimal solution is given by:



(2). (a). We perform elementary row operations on A:

$$A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$$

subtract row 3 from row 4; subtract row 2 from row 3; subtract row 1 from row 2:

$$\longrightarrow \begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \end{bmatrix}$$

subtract row 2 from row 1; subtract row 2 from rows 3 and row 4; $\frac{1}{10}$ row 2

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

subtract row 1 from row 2

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-1 \cdot$ row 2;

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

subtract 2. row 2 from row 1

$$\longrightarrow \begin{array}{c} \begin{array}{cccc} \beta_1 = 1 & \beta_2 = 2 & & \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \Big\} U$$

Thus a basis for range(A) is : $\{a_1, a_2\} = \left\{ \begin{bmatrix} 11 \\ 21 \\ 31 \\ 41 \end{bmatrix}, \begin{bmatrix} 12 \\ 22 \\ 32 \\ 42 \end{bmatrix} \right\}$.

We have:

$$\ker A = \ker U = \left\{ x \in \mathbb{R}^4 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \right\}$$

With $\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and so $z_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

and with $\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and so $z_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

So a basis for $\ker A$ is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2.(b). We have

$$x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 - x_3 x_1 = x^T \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix} x$$

and so the problem can be written as

$$\begin{cases} \text{minimize} & \frac{1}{2} x^T H x + c^T x + c_0 \\ \text{s.t.} & Ax = b \end{cases}$$

where

$$H = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and $c_0 = 0$.

We have

$$x^T H x = \frac{1}{2}\left(2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1\right)$$

$$= \frac{1}{2}\left((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\right)$$

≥ 0, and so H is positive semi definite.

The Lagrange method equations are:

$$H\hat{x} - A^T u = -c \quad \text{and}$$
$$A\hat{x} = b.$$

Clearly

$$A\hat{x} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = b, \quad \text{and so}$$

$\hat{x}$ is feasible.

Moreover,

$$H\hat{x} + c = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= 0 = A^T \cdot 0$$

and so with $u := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$, we also have

$H\hat{x} - A^T u = -c$.

As H is positive semidefinite, we know that $\hat{x}$ is optimal if and only if

$\exists\, u$ s.t. $\begin{cases} H\hat{x} - A^T u = c \quad \text{and} \\ A\hat{x} = b. \end{cases}$

Thus $\hat{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an optimal solution.

The problem can be rewritten as:

(3).(a).
$$\begin{cases} \text{minimize} & -x_1 - 2x_2 + 2x_3 \\ \text{s.t.} & x_1 + 0x_2 + 3x_3 + x_4 = 1 \\ & x_1 + x_2 - x_3 + 0x_4 + x_5 = 1 \\ & x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0,\ x_4 \geqslant 0,\ x_5 \geqslant 0 . \end{cases}$$

Then the problem is in standard form

$$\begin{cases} \min. & c^T x \\ \text{s.t.} & Ax = b \\ & x \geqslant 0 \end{cases}$$

with $c := \begin{bmatrix} -1 \\ -2 \\ +2 \\ 0 \\ 0 \end{bmatrix}$, $A = \left[\begin{array}{ccc|cc} 1 & 0 & 3 & 1 & 0 \\ 1 & +1 & -1 & 0 & 1 \end{array}\right]$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. (Rank $A = 2 = m$).

We begin the simplex method with $\beta = (4, 5)$.
Then $\bar{b} = b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geqslant 0$, and so the

basic solution is feasible.

The simplex multipliers vector y is given by
$$A_\beta^T y = c_\beta$$
i.e., $$y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The reduced costs of the nonbasic variables are given by $r_\nu = c_\nu - A_\nu^T y$, i.e.,
$$r_\nu = \begin{bmatrix} -1 \\ -2 \\ +2 \end{bmatrix} - 0 = \begin{bmatrix} -1 \\ -2 \\ +2 \end{bmatrix}.$$

As $\neg [r_\nu \geqslant 0]$, we can't conclude that the current basic feasible solution is optimal.

Since $r_{\nu_q} = r_{\nu_2} = -2 < 0$, we make $x_{\nu_q} = x_{\nu_2} = x_2$ the new basic variable. We compute $\bar{a}_{\nu_q} = \bar{a}_2$ using $A_\beta \bar{a}_2 = a_2$ i.e., $\bar{a}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then the new basic variable x_2 can increase upto

$$t_{max} = \min \left\{ \frac{\bar{b}_k}{\bar{a}_{\nu_q,k}} : \bar{a}_{\nu_q,k} > 0 \right\}$$

$$= \min \left\{ \frac{1}{1} \right\} = 1 = \frac{\bar{b}_2}{\bar{a}_{2,2}}.$$

So $x_{\beta_p} = x_{\beta_2} = x_5$ leaves the set of basic variables and x_2 takes its place. Hence $\beta = (4,2)$ and $\nu = (1,3,5)$. Thus

$$A_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A_\nu = \begin{bmatrix} 1 & 3 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

We calculate $\bar{b}$ using $A_\beta \bar{b} = b$, i.e.,

$$\bar{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The simplex multipliers vector is obtained by solving $A_\beta^T y = c_\beta$ i.e., $y = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

The reduced costs of the nonbasic variables are given by $r_\nu = c_\nu - A_\nu^T y$

$$= \begin{bmatrix} -1 \\ +2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ +2 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \geqslant 0$$

and so the current basic feasible solution is optimal. So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is optimal for the original problem.

(3). (b). The problem can be rewritten as

$$\begin{cases} \min. & f(x) \\ \text{s.t.} & h(x) = 0 \end{cases}$$

where $f(x) := x_1 - x_2$,

$h(x) := x_1^2 + x_2^2 - 2x_2$.

We have

$$\nabla h(x) = [2x_1 \quad 2x_2 - 2].$$

$\nabla h(x)$ is independent iff $\nabla h(x) \neq 0$.

$\nabla h(x) = 0$ iff $x_1 = 0$, $x_2 = 1$.

But $0^2 + 1^2 - 2 \cdot 1 = -1 \neq 0$.

So $\forall x \in \mathcal{F}_e := \{x \in \mathbb{R}^2 : h(x) = 0\}$, $\nabla h(x) \neq 0$

and so every feasible x is a regular point.

Thus:

If x is an optimal solution,

then $\exists u \in \mathbb{R}$ s.t. $\nabla f(x) + u \nabla h(x) = 0$. $(*)$

We have $\nabla f(x) = [1 \quad -1]$.

So $(*)$ becomes

$$[1 \quad -1] + u\,[2x_1 \quad 2x_2 - 2] = 0$$

i.e., $\begin{cases} 1 + u \cdot 2x_1 = 0 \\ -1 + u \cdot (2x_2 - 2) = 0 \end{cases}$.

Hence $u \neq 0$ and $x_1 = \dfrac{-1}{2u}$

and $x_2 = \dfrac{1}{2u} + 1$.

As $x \in \mathcal{F}_e$, we also have $x_1^2 + x_2^2 - 2x_2 = 0$

i.e., $\dfrac{1}{4u^2} + \dfrac{1}{4u^2} + 1 + \dfrac{1}{u} - \dfrac{2}{2u} - 2 = 0$

i.e., $\dfrac{1}{2u^2} = 1$ and so $u = \pm \dfrac{1}{\sqrt{2}}$.

So $(x_1, x_2) = \left(+\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)$ or $(x_1, x_2) = \left(\frac{-1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$.

We have $x_1 - x_2 = \frac{1}{\sqrt{2}} - \left(1 - \frac{1}{\sqrt{2}}\right) = \sqrt{2} - 1$ in the former case,

and $x_1 - x_2 = -\frac{1}{\sqrt{2}} - 1 - \frac{1}{\sqrt{2}} = -1 - \sqrt{2}$ in the latter.

So the only possibility is $\left(-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$ for the optimal solution.

The feasible set is just the circle with center $(0,1)$ and radius 1:

$\mathcal{F} = \{x : x_1^2 + x_2^2 - 2x_2 = 0\} = \{x \in \mathbb{R}^2 : x_1^2 + (x_2 - 1)^2 = 1\}$.

Thus $\mathcal{F}$ is compact.

The map $x \longmapsto x_1 - x_2$ is continuous.

Thus by the Weierstrass theorem, the optimization problem has an optimal solution.

Hence $\left(-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$ is the unique optimal solution.

(4). (a). We have

$$\frac{\partial f(x)}{\partial x_1} = 2x_1 + 2x_3 + 1$$

$$\frac{\partial f(x)}{\partial x_2} = 4x_2 + 4x_3$$

$$\frac{\partial f(x)}{\partial x_3} = 10x_3 + 2x_1 + 4x_2 + 4x_3^3.$$

Hence

$$\frac{\partial^2 f(x)}{\partial x_1^2} = 2 \qquad \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} = 0 \qquad \frac{\partial^2 f(x)}{\partial x_3 \partial x_1} = 2$$

$$\frac{\partial^2 f(x)}{\partial x_1 \partial x_2} = 0 \qquad \frac{\partial^2 f(x)}{\partial x_2^2} = 4 \qquad \frac{\partial^2 f(x)}{\partial x_3 \partial x_2} = 4$$

$$\frac{\partial^2 f(x)}{\partial x_1 \partial x_3} = 2 \qquad \frac{\partial^2 f(x)}{\partial x_2 \partial x_3} = 4 \qquad \frac{\partial^2 f(x)}{\partial x_3^2} = 10 + 12x_3^2.$$

So the Hessian $F(x)$ of f at x is given by

$$F(x) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 4 & 10+12x_3^2 \end{bmatrix}.$$

The interior of $C = \mathbb{R}^3$ is not empty, and so we know that f is convex iff $F(x)$ is positive semidefinite at all $x \in C = \mathbb{R}^3$. For some elementary matrix E_1 we have

$$E_1 F(x) E_1^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 8+12x_3^2 \end{bmatrix}.$$

For an elementary matrix E_2, we have

$$E_2 E_1 F(x) E_1^T E_2^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4+12x_3^2 \end{bmatrix}.$$

As $x_3^2 \geq 0$, it follows that $4+12x_3^2 > 0$.

Hence $F(x)$ is positive (semi) definite $\forall\, x \in \mathbb{R}^3$.

So f is convex on $\mathbb{R}^3$.

(4) (b) We have

$$\nabla f(0) = \begin{bmatrix} 2x_1+2x_3+1 & 4x_2+4x_3 & 10x_3+2x_1+4x_2+4x_3^3 \end{bmatrix}\Big|_0$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The update equation is

$$F(x^{(1)})\,(x^{(2)} - x^{(1)}) = -(\nabla f(x^{(1)}))^T$$

i.e.,

$$F(0)\,(x^{(2)} - 0) = -(\nabla f(0))^T$$

i.e.,

$$F(0)\, x^{(2)} = -(\nabla f(0))^T.$$

We have

$$F(0) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 4 & 10+12x_3^2 \end{bmatrix}\Bigg|_{x=0}$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 4 & 10 \end{bmatrix}.$$

Hence

$$x^{(2)} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 4 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}.$$

(4) (c). The problem can be rewritten as

$$\begin{cases} \min. & f(x) \\ \text{s.t.} & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \\ & g_3(x) \leq 0 \\ & g_4(x) \leq 0 \\ & g_5(x) \leq 0 \\ & g_6(x) \leq 0 \end{cases}$$

where

$$\begin{aligned} g_1(x) &:= -x_1 \\ g_2(x) &:= -x_2 \\ g_3(x) &:= -x_3 \\ g_4(x) &:= x_1 - 1 \\ g_5(x) &:= x_2 - 1 \\ g_6(x) &:= x_3 - 1. \end{aligned}$$

We have

$$\nabla f(0) = [1 \quad 0 \quad 0]$$

while

$$\begin{aligned} \nabla g_1(0) &= [-1 \quad 0 \quad 0] \\ \nabla g_2(0) &= [0 \quad -1 \quad 0] \\ \nabla g_3(0) &= [0 \quad 0 \quad -1] \\ \nabla g_4(0) &= [1 \quad 0 \quad 0] \\ \nabla g_5(0) &= [0 \quad 1 \quad 0] \\ \nabla g_6(0) &= [0 \quad 0 \quad 1]. \end{aligned}$$

On inspection, with $y_1 := 1$, $y_2 = y_3 = y_4 = y_5 = y_6 = 0$, we have

$$\nabla f(0) + \sum_{i=1}^{6} y_i \nabla g_i(0) = [1 \quad 0 \quad 0] + [-1 \quad 0 \quad 0] = 0,$$

so (KKT-1) is satisfied.

(KKT-2):

$$\begin{aligned} g_1(0) &= 0 \leq 0 & g_4(0) &= -1 \leq 0 \\ g_2(0) &= 0 \leq 0 & g_5(0) &= -1 \leq 0 \\ g_3(0) &= 0 \leq 0 & g_6(0) &= -1 \leq 0 \end{aligned}$$

and so (KKT-2) is also satisfied.

(KKT-3) is satisfied since $y_1 = 1 \geq 0$ and

$y_2 = y_3 = y_4 = y_5 = y_6 = 0 \geq 0$.

(KKT-4) : $y_1 \, g_1(0) = 1 . 0 = 0$

$y_2 \, g_2(0) = 0 . 0 = 0$

$y_3 \, g_3(0) = 0 . 0 = 0$

$y_4 \, g_4(0) = 0 . (-1) = 0$

$y_5 \, g_5(0) = 0 . (-1) = 0$

$y_6 \, g_6(0) = 0 . (-1) = 0$.

So all the KKT-conditions are satisfied.

The problem is clearly convex

(since $f, g_1, \ldots, g_6$ are convex!)

and so $\hat{x} = 0$ is optimal

(since for a convex problem the KKT-conditions

<u>are sufficient</u> for optimality.)

(5.) (a) (i) Let $X = \{x : x_1 \geq 0 \text{ and } x_2 \geq 0\}$.

Define $L : X \times \mathbb{R}^1 \longrightarrow \mathbb{R}$ by

$$L(x,y) = x_1^3 + x_2^3 + y\,(1 - 4x_1 - 9x_2).$$

The relaxed Lagrange problem (PR_y) is the following:

Given $y \geq 0$, minimize $x \mapsto L(x,y)$ on X, i.e.,

$$(PR_y): \begin{cases} \min. & x_1^3 + x_2^3 + y\,(1 - 4x_1 - 9x_2) \\ \text{s.t.} & x_1 \geq 0 \\ & x_2 \geq 0. \end{cases}$$

The function

$$x \mapsto x_1^3 + x_2^3 + y\,(1 - 4x_1 - 9x_2)$$

can be decomposed into the two 1-variable functions

$$x_1 \overset{\varphi_1}{\longmapsto} x_1^3 - 4y\,x_1 \quad \text{and} \quad x_2 \overset{\varphi_2}{\longmapsto} x_2^3 - 9y\,x_2.$$

As $\varphi_1''(x_1) = 6x_1 \geq 0$ for $x_1 \geq 0$ and

$\varphi_2''(x_2) = 6x_2 \geq 0$ for $x_2 \geq 0$,

it follows that φ_1, φ_2 are convex.

They are minimized when

$$3x_1^2 - 4y = 0, \quad 3x_2^2 - 9y = 0, \text{ respectively,}$$

so that

$$\hat{x}_1(y) = \sqrt{\frac{4y}{3}} \geq 0 \quad \text{and} \quad \hat{x}_2(y) = \sqrt{\frac{9y}{3}} \geq 0.$$

Hence the minimizer of (PR_y) is:

$$\hat{x}(y) = \left(\sqrt{\frac{4y}{3}}, \sqrt{\frac{9y}{3}}\right) = \left(2\sqrt{\frac{y}{3}}, 3\sqrt{\frac{y}{3}}\right).$$

(ii) The dual objective function is

$$\varphi(y) = L(\hat{x}(y), y)$$

$$= (2^3+3^3)\left(\sqrt{\frac{y}{3}}\right)^3 + y\left(1 - 2^3\sqrt{\frac{y}{3}} - 3^3\sqrt{\frac{y}{3}}\right)$$

$$= (2^3+3^3)\left(\sqrt{\frac{y}{3}}\right)^3 + y - 3.(2^3+3^3)\left(\sqrt{\frac{y}{3}}\right)^3$$

$$= -2\,(2^3+3^3)\left(\sqrt{\frac{y}{3}}\right)^3 + y.$$

So the dual problem (D) is

$$(D): \begin{cases} \max. & -2(2^3+3^3)\left(\sqrt{\frac{y}{3}}\right)^3 + y \\ \text{s.t.} & y \geq 0. \end{cases}$$

The optimal solution $\hat{y}$ is given by

$$-\cancel{2}.(2^3+3^3).\cancel{3}\left(\sqrt{\frac{y}{3}}\right)^2.\frac{1}{\cancel{2}\sqrt{\frac{y}{3}}}.\frac{1}{\cancel{3}} + 1 = 0$$

$$\sqrt{\frac{y}{3}} = \frac{1}{2^3+3^3}$$

$$\hat{y} = \frac{3}{(2^3+3^3)^2} > 0.$$

(The problem is convex since the second derivative of the objective function is

$$(2^3+3^3).\frac{1}{\sqrt{3}}.\frac{1}{2\sqrt{y}} \geq 0 \text{ for } y > 0.)$$

Hence $\hat{x} = \hat{x}(\hat{y}) = \left(\dfrac{2}{2^3+3^3}, \dfrac{3}{2^3+3^3}\right).$

We have

(1) $\hat{x}$ is feasible for (P):

$$\frac{2}{2^3+3^3} > 0, \quad \frac{3}{2^3+3^3} > 0,$$

$$4 \cdot \frac{2}{2^3+3^3} + 9 \cdot \frac{3}{2^3+3^3} = 1 \geq 1.$$

(2) $\hat{y} = \frac{3}{(2^3+3^3)^2} > 0$ is feasible for (D).

(3)

$$f(\hat{x}) = \left(\frac{2}{2^3+3^3}\right)^3 + \left(\frac{3}{2^3+3^3}\right)^3 = \frac{1}{(2^3+3^3)^2};$$

$$\varphi(\hat{y}) = -2(2^3+3^3)\left(\frac{1}{2^3+3^3}\right)^3 + \frac{3}{(2^3+3^3)^2}$$

$$= \frac{1}{(2^3+3^3)^2}, \text{ and so } f(\hat{x}) = \varphi(\hat{y}).$$

Hence it follows that the global optimality conditions are satisfied, and so $\hat{x} = \left(\frac{2}{2^3+3^3}, \frac{3}{2^3+3^3}\right)$ is optimal for (P).

(5)(b). (i) FALSE.

$\left(\text{Consider } \left\{\begin{array}{ll} \text{min.} & x_1 x_2 \\ \text{s.t.} & \{x \in \mathbb{R}^2 : x_1 = x_2\} \end{array}\right\}.\right)$

(ii) FALSE.

$\left(\text{Consider } \left\{\begin{array}{ll} \text{min.} & x \\ \text{s.t.} & x \in \mathbb{R} \end{array}\right\}.\right)$

(iii) FALSE.

$\left(\text{Consider } \left\{\begin{array}{ll} \text{min.} & 0 \\ \text{s.t.} & x \in \mathbb{R} \end{array}\right\}.\right)$