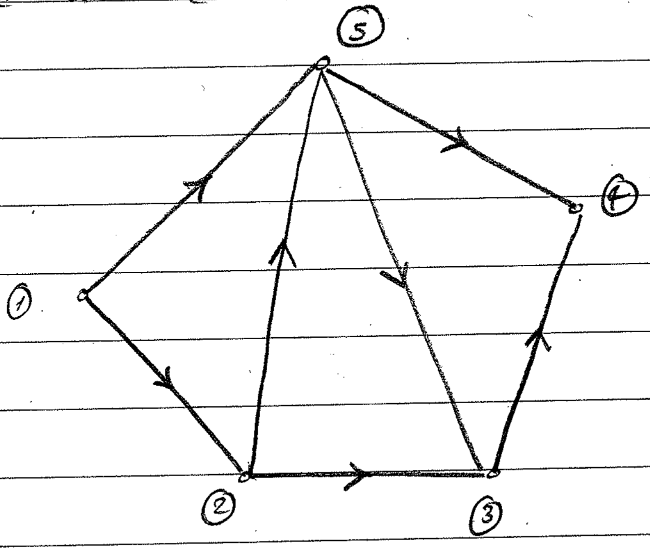
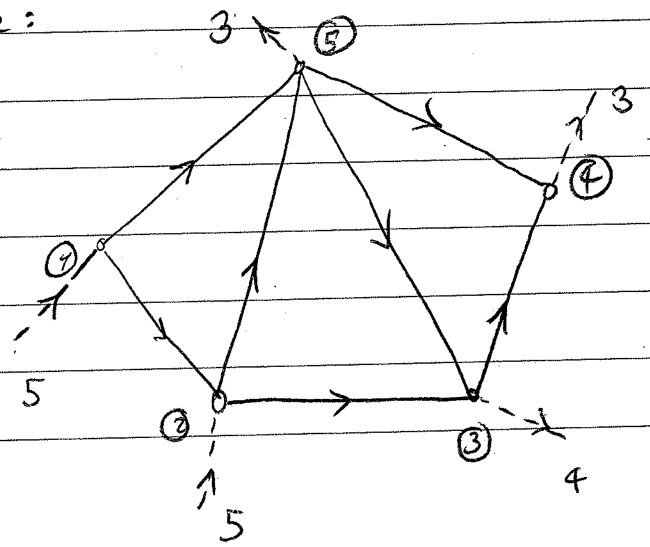
1. (a). As A has 5 rows (which add up to 0), the number of nodes is 5. We have

$$A = \begin{array}{c} \text{nodes} \\ ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{edges}(1,2) & (1,5) & (2,3) & (2,5) & (3,4) & (5,3) & (5,4) \end{array} \\ \left[\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 1 \end{array}\right] \end{array}$$

So we arrive at the following network:



(Also from b, we see that nodes ①, ② are source nodes with supplies 5, 5, respectively, while nodes ③, ④, ⑤ are sink nodes with flows 4, 3, 3, respectively going out of the network. So we have:

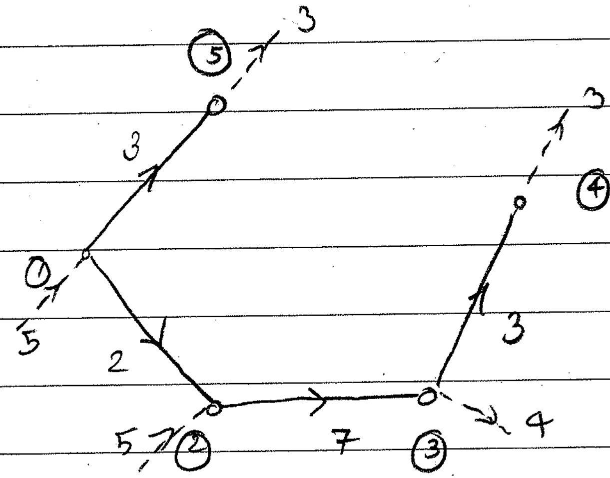


)

The given solution:

$$\hat{x} = \begin{array}{ccccccc} (1,2) & (1,5) & (2,3) & (2,5) & (3,4) & (5,3) & (5,4) \\ \end{array}$$
$$\hat{x} = [\,2 \quad 3 \quad 7 \quad 0 \quad 3 \quad 0 \quad 0\,]^T$$

corresponds to the spanning tree



and it is ≥ 0, and the flow balance is satisfied at each node:

at node ①: $5 = 2 + 3$

" ②: $5 + 2 = 7$

" ③: $7 = 3 + 4$

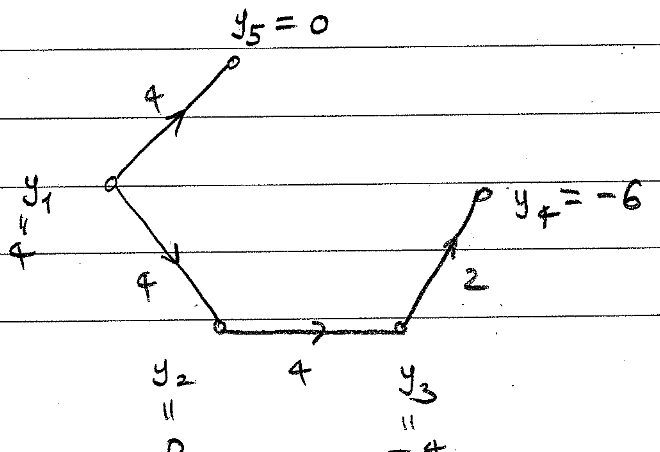
" ④: $3 = 3$

" ⑤: $3 = 3$.

So it is a basic feasible solution.

We now calculate the simplex multipliers vector using: $y_i - y_j = c_{ij}$ for the tree edges (i,j)

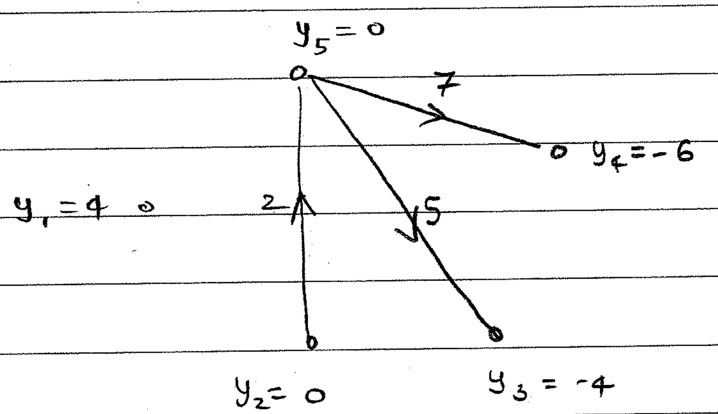
$y_m = 0$.



$$c = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 2 \\ 2 \\ 5 \\ 7 \end{bmatrix} \begin{array}{l} (1,2) \\ (1,5) \\ (2,3) \\ (2,5) \\ (3,4) \\ (5,3) \\ (5,4) \end{array}$$

The reduced costs of the nonbasic variables can be found using:

$$r_{ij} = c_{ij} - (y_i - y_j) \quad \text{for the nontree edges } (i,j)$$



Thus:

$$r_{25} = c_{25} - (y_2 - y_5) = 2 - (0 - 0) = 2 \geq 0$$

$$r_{53} = c_{53} - (y_5 - y_3) = 5 - (0 - (-4)) = 1 \geq 0$$

$$r_{54} = c_{54} - (y_5 - y_4) = 7 - (0 - (-6)) = 1 \geq 0$$

Since all $r_{ij} \geq 0$, we conclude that the current basic feasible solution is optimal

1.(b) For a suitable invertible E_1, we have

$$E_1 H E_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As $1 \geq 0$, $0 \geq 0$, $0 \geq 0$, it follows that H is positive semidefinite.

We have

$$Hc = \begin{bmatrix} 1 & 10 & 100 \\ 10 & 100 & 1000 \\ 100 & 1000 & 10000 \end{bmatrix} \begin{bmatrix} 10 \\ -1 \\ 0 \end{bmatrix}$$

$$= 10 \cdot \begin{bmatrix} 1 \\ 10 \\ 100 \end{bmatrix} + (-1) \begin{bmatrix} 10 \\ 100 \\ 1000 \end{bmatrix} + 0 \begin{bmatrix} 100 \\ 1000 \\ 10000 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 100 \\ 1000 \end{bmatrix} - \begin{bmatrix} 10 \\ 100 \\ 1000 \end{bmatrix} = 0.$$

Thus $c \in \ker H$.

As H is positive semidefinite, we know that (Q) has an optimal solution if and only if $-c \in \operatorname{ran} H$.

But we have seen above that

$$-c \in \ker H = \ker H^T = (\operatorname{ran} H)^\perp$$

As $-c \neq 0$, it follows that $-c \notin \operatorname{ran} H$

(for otherwise $\underbrace{(-c)^T}_{\operatorname{ran} H} \underbrace{(-c)}_{(\operatorname{ran} H)^\perp} = 0$, and so $-c = 0$, which gives a contradiction.)

Thus (Q) has no optimal solution.

2. (a) (i) The problem is in standard form

$$\begin{cases} \min. \ c^T x \\ \text{s.t.} \quad Ax = b \\ \qquad x \geq 0 \end{cases}$$

with $c := \begin{bmatrix} -1 \\ -2 \\ +2 \\ 0 \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 1 & 1 & -1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$.

(Rank $A = 2$.)

We begin with $\beta = (4, 5)$. Then $A_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
$A_\nu = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & -1 \end{bmatrix}$ and $\nu = (1, 2, 3)$.

The corresponding basic solution $\bar{x}$ is obtained by solving for $\bar{b}$ in $A_\beta \bar{b} = b$ and setting $x_\nu = 0$.
$A_\beta \bar{b} = b$ becomes $I_2 \bar{b} = b$ and so $\bar{b} = b = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$.
Hence $\bar{x}_\beta = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \bar{b} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$, while

$$\bar{x}_\nu = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

As $\bar{b} \geq 0$, this basic solution is feasible.
So the initial basic feasible solution is

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \\ 7 \end{bmatrix}$$

To investigate optimality, we calculate the simplex multipliers vector y and the reduced costs of the nonbasic variables.

The simplex multipliers vector y is given by $A_\beta^T y = c_\beta$

i.e., $y = c_\beta = \begin{bmatrix} c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The reduced costs of the nonbasic variables are given by

$$r_\nu = c_\nu - A_\nu^T y$$

$$= \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} - A_\nu^T 0 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ +2 \end{bmatrix} \begin{array}{l} q=1 \\ q=2 \\ q=3 \end{array}$$

As $\neg[r_\nu \geq 0]$, we cannot conclude that $\bar{x}$ is optimal.

As $r_{\nu_q} = r_{\nu_2} = -2 < 0$, we make $x_{\nu_q} = x_{\nu_2} = x_2$ the new basic variable.

We compute $\bar{a}_{\nu_q} = \bar{a}_2$ using $A_\beta \bar{a}_2 = a_2$

and so $\bar{a}_2 = a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

As $\neg[\bar{a}_2 \leq 0]$, the new basic variable x_2 can increase upto

$$t_{max} = \min \left\{ \frac{(\bar{b})_k}{(\bar{a}_{\nu_q})_k} : (\bar{a}_{\nu_q})_k > 0 \right\}$$

$$= \min \left\{ \frac{7}{1} \right\} = \frac{(\bar{b})_2}{(\bar{a}_2)_2}.$$

So $p = 2$ and $\beta_p = \beta_2 = 5$ leaves the basic tuple.

Hence $\beta_{new} = (4, \boxed{2})$ is the new basic tuple and

$\nu_q = 2$, $\beta_p = 5$

$\nu_{new} = (1, \boxed{5}, 3)$ is the new nonbasic tuple.

$\beta_p = 5$

Now

$$A_\beta = \begin{array}{cc} a_4 & a_2 \\ \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_\nu = \begin{array}{ccc} a_1 & a_5 & a_3 \end{array} \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & -1 \end{bmatrix}.$$

We calculate $\bar{b}$ using $A_\beta \bar{b} = b$ and so $I_2 \bar{b} = b$ i.e., $\bar{b} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$.

The simplex multipliers vector is obtained by solving $A_\beta^T y = c_\beta$ i.e., $I_2 y = c_\beta$ and so

$$y = c_\beta = \begin{bmatrix} c_4 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

The reduced costs of the non-basic variables is given by

$$r_\nu = c_\nu - A_\nu^T y$$
$$= \begin{bmatrix} c_1 \\ c_5 \\ c_3 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ 0 \\ +2 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

As $r_\nu \geq 0$, the current basic feasible solution, namely

$$\hat{x} = \begin{bmatrix} 0 \\ 7 \\ 0 \\ 7 \\ 0 \end{bmatrix}$$

is optimal for (P)

2.(a).(ii) The dual to

$$\begin{cases} \min. & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$$

is given by

$$\begin{cases} \max. & b^T y \\ \text{s.t.} & A^T y \leq c \end{cases}$$

In our case, this is

$$\begin{cases} \text{maximize} & 7y_1 + 7y_2 \\ \text{s.t.} & y_1 + y_2 \leq -1 \\ & y_2 \leq -2 \\ & 3y_1 - y_2 \leq 2 \\ & y_1 \leq 0 \\ & y_2 \leq 0 \end{cases}$$

We know that the last simplex multipliers vectors corresponding to an optimal solution $\hat{x}$ to (LP) is optimal for the dual problem. Thus

$$y = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$ is optimal for the dual problem

2. (b). (i) FALSE.

Consider $\begin{cases} \min & x_1 \\ \text{s.t.} & x_1 + 2x_2 = 0 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{cases}$

Then $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the basic feasible solution corresponding to $\beta = (1)$, but has all components 0

(ii) FALSE.

In the same example in (i),

we have $y = (A_\beta^T)^{-1} c_\beta = 1 . c_1 = +1$

and $r_\nu = c_\nu - A_\nu^T y$

$= +0 - 2 . 1$

$= -2 < 0$

but clearly $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is optimal.

(iii) TRUE.

After having started with a basic feasible solution, we are guaranteed to get an x which is a basic feasible solution after every iteration of the simplex method.

(iv) FALSE.

Consider $\begin{cases} \min & x_1 \\ \text{s.t.} & x_1 - x_2 = 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{cases}$

There are two basic tuples: $\beta = (1)$ and $\beta = (2)$, but only one basic feasible solution,

namely $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (corresponding to $\beta = (1)$).

3. (a). We have

$$f(x) := 5x_1^2 + 4x_1x_2 + x_2^2$$

$$= \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{2} x^T H x,$$

where

$$H := \begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}.$$

For a suitable invertible E_1, we have

$$E_1 H E_1^T = \begin{bmatrix} 10 & 0 \\ 0 & 2 - \frac{4}{10}\cdot 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & \frac{4}{10} \end{bmatrix}$$

is positive definite. So H is positive definite

$A = \begin{bmatrix} 3 & 2 \end{bmatrix}$

and so a basis for $\ker A$ is given by

$$Z := \begin{bmatrix} -2/3 \\ 1 \end{bmatrix}.$$

As H is positive semidefinite, so is $Z^T H Z$.

Hence we have:

$$\hat{x} \text{ is optimal } \iff \begin{cases} \exists\, \hat{v} \text{ s.t. } Z^T H Z \hat{v} = -Z^T(H\bar{x} + c) \\ \text{and } \hat{x} = \bar{x} + Z\hat{v}. \end{cases}$$

We take $\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $A\bar{x} = 3 \cdot 1 + 2 \cdot 1 = 5 = b$.

We have:

$$Z^T H Z = \begin{bmatrix} -2/3 & 1 \end{bmatrix}\begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 & 1 \end{bmatrix}\begin{bmatrix} -\frac{20}{3} + 4 \\ -\frac{8}{3} + 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 & 1 \end{bmatrix}\begin{bmatrix} -8/3 \\ -2/3 \end{bmatrix} = \frac{16}{9} - \frac{2}{3} = \frac{16}{9} - \frac{6}{9} = \frac{10}{9},$$

$$+Z^T(H\bar{x} + c) \underset{(c=0)}{=} \begin{bmatrix} -2/3 & 1 \end{bmatrix}\begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 & 1 \end{bmatrix}\begin{bmatrix} 14 \\ 6 \end{bmatrix} = -\frac{28}{3} + 6$$

$$= -\frac{10}{3}.$$

Thus $\hat{v} = \frac{9}{16} \cdot \left(\frac{10}{3}\right) = 3$.

Hence $\hat{x} = \bar{x} + Zv$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} \cdot 3$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

is the (unique) optimal solution.

3.(b). We have

$$f'(x) = \frac{1}{1+e^{-2x}} \cdot (-2)\, e^{-2x} + 1$$

$$= \frac{-2e^{-2x}}{1+e^{-2x}} + 1,$$

and so

$$f''(x) = \frac{-2.(-2)e^{-2x}(1+e^{-2x}) - (-2e^{-2x})(-2.e^{-2x})}{(1+e^{-2x})^2}$$

$$= \frac{4e^{-2x}(1+e^{-2x}) - 4e^{-4x}}{(1+e^{-2x})^2}$$

$$= \frac{4e^{-2x} + \cancel{4e^{-4x}} - \cancel{4e^{-4x}}}{(1+e^{-2x})^2}$$

$$= \frac{4e^{-2x}}{(1+e^{-2x})^2} \geq 0 \qquad (\because e^r > 0 \quad \forall\ r \in \mathbb{R}).$$

Thus Newton's method gives

$$F(x^{(k)})\,(x^{(k+1)} - x^{(k)}) = -(\nabla f(x^{(k)}))^T$$

i.e.,

$$\frac{4e^{-2x^{(k)}}}{(1+e^{-2x^{(k)}})^2} \cdot (x^{(k+1)} - x^{(k)}) = + \frac{2e^{-2x^{(k)}}}{1+e^{-2x^{(k)}}} - 1$$

i.e.,

$$\frac{4e^{-2x^{(k)}}}{(1+e^{-2x^{(k)}})^{\cancel{2}}} (x^{(k+1)} - x^{(k)}) = \frac{e^{-2x^{(k)}} - 1}{\cancel{1+e^{-2x^{(k)}}}}$$

i.e.,

$$x^{(k+1)} = x^{(k)} + \frac{(e^{-2x^{(k)}} - 1)(e^{-2x^{(k)}} + 1)}{4e^{-2x^{(k)}}}$$

$$= x^{(k)} + \frac{e^{-4x^{(k)}} - 1}{4e^{-2x^{(k)}}}$$

$$= x^{(k)} + \frac{1}{2} \cdot \frac{e^{-2x^{(k)}} - e^{+2x^{(k)}}}{2} = x^{(k)} + \frac{1}{2} \cdot \left(-\sinh(2x^{(k)})\right)$$

$$= x^{(k)} - \frac{1}{2} \sinh(2x^{(k)}).$$

3.4

If $x^{(k)} \xrightarrow{k\to\infty} L$, then $L = L - \frac{1}{2}\sinh(2L)$, i.e.,
$\sinh(2L) = 0$, i.e., $\underline{e^{2L} - e^{-2L}} = 0$ and so $e^{4L} = 1$.
Thus $4L = 0$ and so $\overset{2}{L} = 0$.

3. (c). A subset C of $\mathbb{R}^n$ is said to be <u>convex</u> if for all $x \in C$, all $y \in C$ and all $t \in (0,1)$, $(1-t)x + ty \in C$.

Consider the function $g : \mathbb{R}^3 \to \mathbb{R}$ given by

$$g(x) = x_1^8 + x_2^{12} + x_3^{16} - 2011.$$

We have

$$\nabla g(x) = \begin{bmatrix} 8x_1^7 & 12x_2^{11} & 16x_3^{15} \end{bmatrix}$$

and so its Hessian is given by

$$G(x) = \begin{bmatrix} 8.7.x_1^6 & 0 & 0 \\ 0 & 12 \cdot 11 x_2^{10} & 0 \\ 0 & 0 & 16 \cdot 15 x_3^{14} \end{bmatrix}$$

which is positive semidefinite for each $x \in \mathbb{R}^3$. As $\mathbb{R}^3$ has interior points, it follows that g is a convex function.

Hence the set

$$\{x \in \mathbb{R}^3 : g(x) \le 0\} \text{ is convex,}$$

$$\parallel$$

$$\left(\{x \in \mathbb{R}^3 : x_1^8 + x_2^{12} + x_3^{16} \le 2011\} \right)$$

4. (a) Let $p(t) = (t, t, t)$, $t \in \mathbb{R}$.

Then

$$f(p(t)) = t^2 + t^2 + t^2 - 2 \cdot t \cdot t \cdot t$$
$$= 3t^2 - 2t^3$$
$$= t^3 \left(\frac{3}{t} - 2 \right).$$

As $t \longrightarrow \infty$, $\frac{3}{t} - 2 \longrightarrow -2$, and so $f(p(t)) \longrightarrow -\infty$.

So clearly the set of values of f on $\mathbb{R}^3$ is not bounded below.

We have

$$\nabla f(x) = \begin{bmatrix} 2x_1 - 2x_2 x_3 & 2x_2 - 2x_1 x_3 & 2x_3 - 2x_1 x_2 \end{bmatrix}$$

and so

$$\nabla f(v) = \begin{bmatrix} 2 \cdot 1 - 2 \cdot 1 \cdot 1 & 2 \cdot 1 - 2 \cdot 1 \cdot 1 & 2 \cdot 1 - 2 \cdot 1 \cdot 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \underline{0}.$$

The Hessian $F(x)$ of f at x is given by

$$F(x) = \begin{bmatrix} 2 & -2x_3 & -2x_2 \\ -2x_3 & 2 & -2x_1 \\ -2x_2 & -2x_1 & 2 \end{bmatrix}$$

Hence $F(v) = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$. For a suitable invertible E_1, we have $E_1 F(v) E_1^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{bmatrix}$ ← non-zero

So $F(v)$ is not positive semidefinite. Thus v is not a local minimizer. (since for v to be a local minimizer, it is necessary that $F(v)$ is p.s.d.)

4. (b). The problem can be rephrased as

$$\begin{cases} \min. & f(x) \\ s.t. & h(x) = 0 \end{cases}$$

, where $f(x) := (x_1 - 2)^2 + x_2^2$,

$h(x) := 4x_1^2 + x_2^2 - 4$

We have

$$\nabla h(x) = [2x_1 \quad 2x_2]$$

$\nabla h(x)$ is independent if and only if $\nabla h(x) \neq 0$.

If $x \in \mathcal{X} := \{x \in \mathbb{R}^2 : h(x) = 0\}$, then $(x_1, x_2) \neq (0, 0)$. So $\nabla h(x) = [2x_1 \quad 2x_2] \neq [0 \quad 0]$. $\forall x \in \mathcal{X}$.

Hence every feasible x is a regular point.

Thus:

if x is an optimal solution,

then $\exists u \in \mathbb{R}$ s.t. $\nabla f(x) + u \nabla h(x) = 0$ $(*)$

We have $\nabla f(x) = [2(x_1 - 2) \quad 2x_2]$.

So $(*)$ becomes

$$[2(x_1 - 2) \quad 2x_2] + u[2x_1 \quad 2x_2] = [0 \quad 0],$$

i.e.,

$$\begin{cases} 2(x_1 - 2) + u \cdot 2x_1 = 0 \\ 2x_2 + u \cdot 2x_2 = 0 \end{cases}$$

Hence

$$\begin{cases} (1 + 4u)\, x_1 = 2 & (**) \\ (1 + u)\, x_2 = 0 & (***) \end{cases}$$

From $(**)$, we see that $1 + u \neq 0$.

So $(***)$ implies that $x_2 = 0$.

Finally from $x_1^2 + x_2^2 = 1$,

we obtain that $x_1 = +1$ or -1.

So possible optimal solutions are $(1,0)$ and $(-1,0)$.

Also $f(1,0) = 1 < f(-1,0) = 9$. So the only possibility for an optimal solution is $(1,0)$.

The feasible set $\mathcal{F}$ is bounded (indeed, $\mathcal{F}$ is contained in the ball with center 0 and radius 1), and it is also closed. So $\mathcal{F}$ is compact.
The map $x \overset{f}{\longmapsto} (x_1-2)^2 + x_2^2$ is continuous.
So we know that $f : \mathcal{F} \longrightarrow \mathbb{R}$ has a global minimizer on $\mathcal{F}$, by the Weierstrass Theorem.

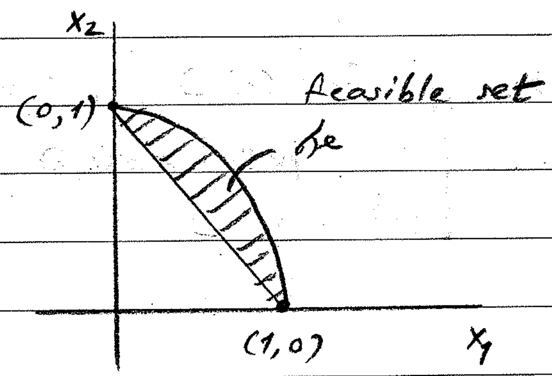
Hence $(1,0)$ is the unique optimal solution.

5. (a). The problem is of the form

$$\begin{cases} \min. & f(x) \\ s.t. & g_1(x) \le 0 \\ & g_2(x) \le 0, \end{cases}$$

where

$$f(x) := x_1 x_2$$
$$g_1(x) := x_1^2 + x_2^2 - 1$$
$$g_2(x) := 1 - x_1 - x_2.$$

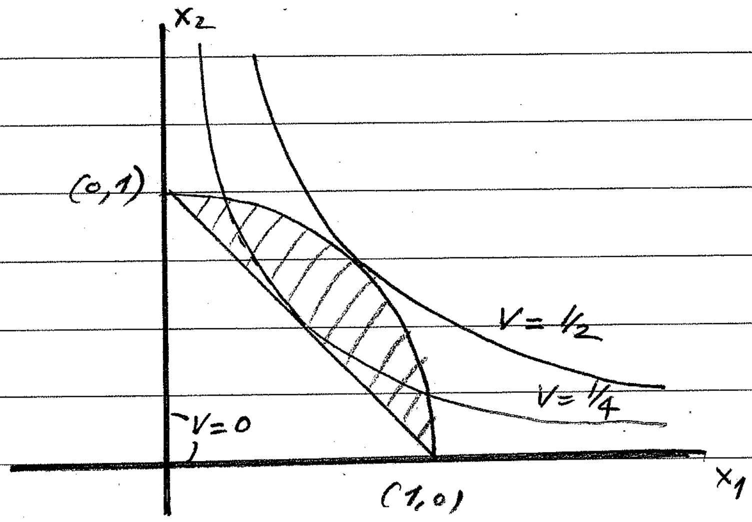


We have $\nabla f(x) = [\, x_2 \quad x_1 \,]$

and $F(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

As the Hessian $F(x)$ of f at x is not positive semidefinite and since C has interior points, it follows that f is not convex. So the problem is <u>not</u> a convex optimization problem.

<u>Sketch of level sets:</u>



KKT-conditions: $\exists y \in \mathbb{R}^2$ s.t.

(KKT-1): $\nabla f(x) + y^T \nabla g(x) = 0$ i.e.,

$$[x_2 \quad x_1] + [y_1 \quad y_2]\begin{bmatrix} 2x_1 & 2x_2 \\ -1 & -1 \end{bmatrix} = [0 \quad 0].$$

i.e., $\begin{cases} x_2 + y_1 . 2x_1 - y_2 = 0 \\ x_1 + y_1 . 2x_2 - y_2 = 0 \end{cases}$

(KKT-2). $g_i(x) \leq 0 \quad \forall\ i = 1, \ldots, m$, i.e.,

$$\begin{cases} x_1^2 + x_2^2 \leq 1 \\ x_1 + x_2 \geqslant 1 \end{cases}$$

(KKT-3). $y \geq 0$ i.e., $\begin{cases} y_1 \geqslant 0 \\ y_2 \geqslant 0 \end{cases}$

(KKT-4) $y_i g_i(x) = 0 \quad \forall\ i = 1, \ldots, m$, i.e.,

$$y_1 . (x_1^2 + x_2^2 - 1) = 0$$
$$y_2 . (1 - x_1 - x_2) = 0.$$

With $\hat{x} := (1, 0)$, we have

$$x_1^2 + x_2^2 = 1^2 + 0^2 = 1 \leq 1 \quad \text{and}$$
$$x_1 + x_2 = 1 + 0 = 1 \geqslant 1,$$

and so (KKT-2) is satisfied.

Also, $y_1 (x_1^2 + x_2^2 - 1) = y_1 (1^2 + 0^2 - 1) = y_1 . 0 = 0$, and

$y_2 (1 - x_1 - x_2) = y_2 (1 - 1 - 0) = y_2 . 0 = 0$,

and so (KKT-3) is satisfied (with every choice of $y_1, y_2 \in \mathbb{R}$

(KKT-1) becomes: $\left\{ \begin{matrix} 0 + y_1 . 2.1 - y_2 = 0 \\ 1 + y_1 . 2.0 - y_2 = 0 \end{matrix} \right\}$ i.e., $\left\{ \begin{matrix} 2y_1 = y_2 \\ y_2 = 1 \end{matrix} \right\}$

and so $y_2 = 1$ and $y_1 = 1/2$.

So (KKT-1) is satisfied with $y = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

Also $y \geqslant 0$ and so (KKT-3) is satisfied.

So for $\hat{x} = (1,0)$, the KKT-conditions are satisfied with $y = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

We have $f(1,0) = 1.0 = 0$
while from the sketch of the feasible set, it is clear that if $x \in \mathcal{F}_e$, then $x_1 \geqslant 0$ and $x_2 \geqslant 0$.
In particular, for $x \in \mathcal{F}_e$,

$$f(x) = \underbrace{x_1}_{\geqslant 0} \underbrace{x_2}_{\geqslant 0} \geqslant 0 = f(1,0).$$

So $(1,0)$ is an optimal solution.

5. (b). Let $X = \mathbb{R}$. Define $L : X \times \mathbb{R} \to \mathbb{R}$ by

$$L(x, y) = x^2 + y(1-x).$$

The relaxed Lagrange problem is the following:

Given $y \geq 0$, minimize $x \mapsto L(x,y)$ on X, i.e.,

$$(PR_y) : \begin{cases} \min. & x^2 + y(1-x) \\ s.t. & x \in \mathbb{R} \end{cases}$$

We have $\frac{d}{dx}\left(x^2 + y(1-x)\right) = 2x - y = 0 \iff x = \frac{y}{2}$,

and $\frac{d^2}{dx^2}\left(x^2 + y(1-x)\right) = 2 > 0 \ \forall x \in \mathbb{R}$.

So the convex problem (PR_y) has the unique optimal solution $\hat{x}(y) := \frac{y}{2}$.

The dual objective function is

$$\begin{aligned} \varphi(y) &= L(\hat{x}(y), y) \\ &= \left(\frac{y}{2}\right)^2 + y\left(1 - \frac{y}{2}\right) \\ &= \frac{y^2}{4} + y - \frac{y^2}{2} = y - \frac{y^2}{4}. \end{aligned}$$

So the dual to (P) is

$$(D) : \begin{cases} \text{maximize} & y - \frac{y^2}{4} \\ s.t. & y \geq 0. \end{cases}$$

We have $\frac{d}{dy}\left(y - \frac{y^2}{4}\right) = 1 - \frac{2y}{4} = 1 - \frac{y}{2} = 0 \iff y = 2$.

Also, $\frac{d^2}{dy^2}\left(y - \frac{y^2}{4}\right) = -\frac{1}{2} < 0$.

So $y = 2$ is a global maximizer of $y \mapsto y - \frac{y^2}{4} : \mathbb{R} \to \mathbb{R}$

In particular, also (as $\hat{y} = 2 \geq 0$) it is a global minimizer of $y \mapsto y - \frac{y^2}{4} : [0, \infty) \to \mathbb{R}$

Hence $\hat{y} := 2$ is the unique optimal solution to (D).

Let $\hat{x} := \hat{x}(\hat{y}) = \frac{\hat{y}}{2} = \frac{2}{2} = 1$.

Then $\hat{x}$ is feasible for (P) since $\hat{x} = 1 \geq 1$.

Also

$$f(\hat{x}) = \hat{x}^2 = 1^2 = 1$$

and $\varphi(\hat{y}) = \hat{y} - \frac{\hat{y}^2}{4} = 2 - \frac{4}{4} = 2 - 1 = 1$.

Finally $\hat{y} = 2 \geq 0$

So $\hat{y} := 2$ is optimal for (D)
and $\hat{x} := 1$ is optimal for (P).