KTH Matematik

Exam in SF1811/SF1841 Optimization.
April 7, 2015, 8:00–13:00

Examiner: Krister Svanberg, telephone: 790 7137, email: krille@math.kth.se.

Allowed utensils: Pen, paper, eraser and ruler. (Penna, papper, suddgummi och linjal.)
No calculator! (Ingen räknare!) A formula-sheet is handed out.

Language: Your solutions should be written in English or in Swedish.

Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.
A passing grade E is guarranteed for 25 points, including bonus points from the home assignments during Nov-Dec 2014. 23-24 points give a possibility to complement the exam to grade E within three weeks from the announcement of the results. Contact the examiner as soon as possible by email if this is the case.
Write your name on each page of the solutions you hand in and number the pages.
Write the solutions to the different questions 1,2,3,4,5 on separate sheets.
This is important since the exams are split up during the corrections.

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1. Consider the LP problem

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \\ -7 \\ -8 \end{pmatrix}, \quad \mathbf{c}^{\mathsf{T}} = (5,\ 7,\ 1,\ 4,\ 2,\ 6,\ 3).$$

(a) As can be seen from the special structure of $\mathbf{A}$, this is in fact a minimum cost network flow problem. Illustrate the corresponding network in a figure. . (2p)

(b) Show that $\hat{\mathbf{x}} = (4, 0, 9, 0, 15, 0, 8)^{\mathsf{T}}$ is an optimal solution to the problem. (3p)

(c) Formulate, in details, the corresponding dual problem D (no matrices in the answer), and give an optimal solution to D. Also check that the complementarity conditions are satisfied. .... (3p)

(d) Verify that the spanning tree corresponding to the above optimal solution still corresponds to an optimal solution also if the right hand side vector $\mathbf{b}$ is changed from $(4,\ 5,\ 6, -7, -8)^{\mathsf{T}}$ to $(7,\ 8, -4, -5, -6)^{\mathsf{T}}$. What are the new primal and dual optimal solutions and optimal values after this change of $\mathbf{b}$? .... (3p)

2. This exercise deals with the shortest distance beween two given lines in $I\!R^3$.

Let the line L_1 be the intersection of the two (non-parallel) planes
$x_1 - x_2 + x_3 = 1$ and $x_1 + x_2 - x_3 = 1$,
and let the line L_2 be the intersection of the two (non-parallel) planes
$x_1 - x_2 - x_3 = 1$ and $-x_1 + x_2 - x_3 = 1$.
Calculate the shortest distance between L_1 and L_2. Also calculate the two points (one on L_1 and the other on L_2) between which the distance is the shortest. . (9p)

Hint:
A line L in $I\!R^3$ can always be written on so called *parmeter form*, i.e. on the form $L = \{\, \mathbf{x} \in I\!R^3 \mid \mathbf{x} = \mathbf{x}_0 + t \cdot \mathbf{d},\ t \in I\!R \,\}$, where $\mathbf{x}_0$ and $\mathbf{d} \neq \mathbf{0}$ are two vectors in $I\!R^3$ and $t \in I\!R$ is the parameter.

3. Consider the function $f : I\!R^2 \to I\!R$ defined by

$$f(\mathbf{x}) = x_1^3 - 3x_1 + x_1x_2 + \tfrac{1}{2}x_1^2x_2^2, \quad \text{where } \mathbf{x} = (x_1, x_2)^\mathsf{T}.$$

Consider the following three points: $\mathbf{x} = (0, 3)^\mathsf{T}$, $\mathbf{x} = (1, -1)^\mathsf{T}$ and $\mathbf{x} = (-1, 1)^\mathsf{T}$.

(a) How many (if any) of these points are local optimal solutions to the problem of minimizing $f(\mathbf{x})$ without any constraints? Motivate your answer. .....(3p)

(b) Perform a complete iteration with *Newtons method* for minimizing $f(\mathbf{x})$ without any constraint, starting from the point $\mathbf{x} = (1,\ 0)^\mathsf{T}$. ....(6p)

(c) Is $\mathbf{x} = (1,\ 0)^\mathsf{T}$ a global optimal solution to the problem of minimizing $f(\mathbf{x})$ subject to the constraints $x_1 \geq 0$ and $x_2 \geq 0$? Motivate carefully. ....(2p)

4. Consider the following LP problem with 21 variables.

$$\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{j=1}^{21} |j-11|\, x_j & (\ 10x_1 + 9x_2 + \ldots + 9x_{20} + 10x_{21}) \\
\text{subject to} & \displaystyle\sum_{j=1}^{21} (21-j)\, x_j = 40, & (\ 20x_1 + 19x_2 + \ldots + x_{20} = 40) \\
& \displaystyle\sum_{j=1}^{21} (j-1)\, x_j = 20, & (\ \ x_2 + 2x_3 + \ldots + 20x_{21} = 20) \\
& x_j \geq 0,\ j = 1, \ldots, 21. &
\end{array}$$

(a) Start with the variables x_1 och x_{21} as basic variables and perform a complete iteration with the Simplex method. Is the new basic solution you obtain an optimal solution to the problem? Motivate. ....(6p)

(b) How many different *feasible basic solutions* with x_1 as one of the basic variables are there to this problem? And how many of these are *optimal* solutions? (4p)

5. Let $\mathbf{p}_1, \ldots, \mathbf{p}_m$ be m given points in $I\!R^n$ and consider the following optimization problem in the variables $z \in I\!R$ and $\mathbf{x} \in I\!R^n$:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}z^2 \\ \text{subject to} & \|\,\mathbf{x}-\mathbf{p}_i\|^2 - z \le 0\,, \;\; i = 1, \ldots, m. \end{array}$$

where, as usual, $\|\,\mathbf{x}-\mathbf{p}_i\,\|^2 = (\mathbf{x}-\mathbf{p}_i)^\mathsf{T}(\mathbf{x}-\mathbf{p}_i)$.

(a) Use *Lagrange relaxation* to deduce an *explicit* expression (containing $\mathbf{y}$ and the m points $\mathbf{p}_i$) for the dual objective function $\varphi(\mathbf{y})$, valid for all dual variable vectors $\mathbf{y} \in I\!R^m$ with $\mathbf{y} \ge \mathbf{0}$. .... (5p)
Hint:
Shorter expressions are obtained by introducing the matrix $\mathbf{P}$ with columns $\mathbf{p}_1, \ldots, \mathbf{p}_m$, and the two vectors $\mathbf{q} = (\|\mathbf{p}_1\|^2, \ldots, \|\mathbf{p}_m\|^2)^\mathsf{T}$ and $\mathbf{e} = (1, \ldots, 1)^\mathsf{T}$.

(b) Assume now that $m = n = 2$, $\mathbf{p}_1 = (1,\,0)^\mathsf{T}$ and $\mathbf{p}_2 = (0,\,1)^\mathsf{T}$.
Let $\hat{\mathbf{y}} = (0.25,\,0.25)^\mathsf{T}$ and calculate the corresponding solution $(\hat{z}, \hat{\mathbf{x}})$ to the original (primal) problem. Then show that $\hat{\mathbf{y}}$ is an optimal solution to the dual problem, while $(\hat{z}, \hat{\mathbf{x}})$ is an optimal solution to the primal problem. ...... (4p)

Note: An interpretation of the problem in (a), which should not be used for solving the problem in (b), is the following: Assume that $(\hat{z}, \hat{\mathbf{x}})$ is an optimal solution to the problem in (a). Then the square root of $\hat{z}$ is the radius of the smallest sphere in $I\!R^n$ which contains all the given points $\mathbf{p}_i$, while $\hat{\mathbf{x}}$ is the center of this sphere.

Good luck!