KTH Matematik

Exam in SF1811/SF1841 Optimization.
Saturday, January 18, 9:00–14:00

Examiner: Krister Svanberg, telephone: 790 7137, email: krille@math.kth.se.

Allowed utensils: Pen, paper, eraser and ruler. (Penna, papper, suddgummi och linjal.)
No calculator! (Ingen räknare!) A formula-sheet is handed out.

Language: Your solutions should be written in English or in Swedish.

Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.
A passing grade E is guarranteed for 25 points, including bonus points from the home assignments during Nov-Dec 2013. 23-24 points give a possibility to complement the exam to grade E within three weeks from the announcement of the results. Contact the examiner as soon as possible by email if this is the case.
Write your name on each page of the solutions you hand in and number the pages.
Write the solutions to the different questions 1,2,3,4,5 on separate sheets.
(This is important since the exams are split up during the corrections.)

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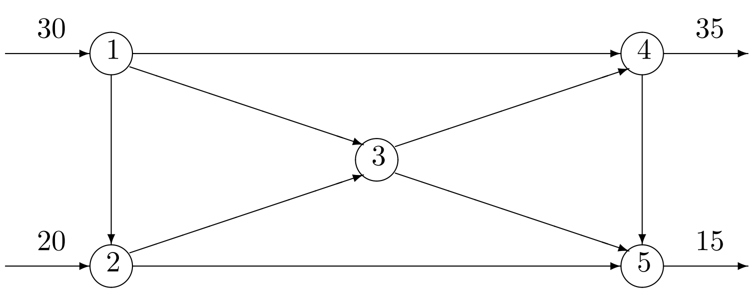
1. In the network below, nodes 1 and 2 are source nodes with supplies 30 and 20 units, nodes 4 and 5 are sink nodes with demands 35 and 15, while node 3 is an intermediate node without any supply or demand.
The costs per unit flow in the various arcs are given by

$c_{12} = c_{23} = c_{34} = c_{45} = 1$, $c_{13} = c_{14} = c_{25} = c_{35} = k$, where k is a constant > 1.



(a) The minimum cost flow problem corresponding to these assumptions can be formulated as an LP problem on the form:

$$\text{minimize } \mathbf{c}^\mathsf{T}\mathbf{x} \text{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}.$$

Write down, in details, the matrix $\mathbf{A}$ and the vectors $\mathbf{b}$ and $\mathbf{c}$ for this example, and describe what the components in the variable vector $\mathbf{x}$ stand for. . . . (3p)

(b) The four arcs with $c_{ij} = 1$ corresponds to a spanning tree, so a basic solution to the problem is obtained by sending suitable flows in just these four arcs. Calculate the arc flows for this basic solution, and decide if it is a *feasible* basic solution or not. . . . . (2p)

(c) For which values of the constant k is the basic solution above an optimal solution to the considered minimum cost problem?
(Hint: Calculate the reduced cost for each of the four non-basic arcs.) . . . (3p)

2. (a) Consider the following LP problem on standard form.

$$\begin{array}{lrl} \text{P1:} & \text{minimize} & x_1 + 3x_2 + x_3 + x_4 \\ & \text{subject to} & x_1 - x_2 + x_3 - x_4 = 2, \\ & & x_1 + x_2 - x_3 - x_4 = 4, \\ & & x_j \geq 0, \quad j = 1, 2, 3, 4. \end{array}$$

Show that $(x_1,\, x_2,\, x_3,\, x_4) = (3,\, 1,\, 0,\, 0)$ is an optimal solution to P1. . . . (3p)

(b) Formulate the dual LP problem D1 corresponding to the primal problem P1. Illustrate D1 in a figure containing the feasible region of D1 and at least two level sets for the objective function in D1. Determine, *using your figure*, the optimal solution to D1, and check that the optimal values of P1 and D1 are equal. . . . . (3p)

(c) Assume now that both the equality constraints in P1 are changed to inequality constraints of the type $\geq$, so that the following problem P2 is obtained:

$$\begin{array}{lrl} \text{P2:} & \text{minimize} & x_1 + 3x_2 + x_3 + x_4 \\ & \text{subject to} & x_1 - x_2 + x_3 - x_4 \geq 2, \\ & & x_1 + x_2 - x_3 - x_4 \geq 4, \\ & & x_j \geq 0, \quad j = 1, 2, 3, 4. \end{array}$$

Use the simplex method to calculate an optimal solution to P2.
You may use the results from (a) above to get a feasible basic solution to start the simplex method from when solving P2. . . . . (3p)

(d) Assume now that both the equality constraints in P1 are changed to inequality constraints of the type $\leq$, so that the following problem P3 is obtained:

$$\begin{array}{lrl} \text{P3:} & \text{minimize} & x_1 + 3x_2 + x_3 + x_4 \\ & \text{subject to} & x_1 - x_2 + x_3 - x_4 \leq 2, \\ & & x_1 + x_2 - x_3 - x_4 \leq 4, \\ & & x_j \geq 0, \quad j = 1, 2, 3, 4. \end{array}$$

Use the simplex method to calculate an optimal solution to P3.
This time, it is recommended that you start with the slack variables as basic variables. . . . . (3p)

3. Consider the following equality-constrained quadratic programming problem QP1:

$$\begin{array}{lll} \text{QP1:} & \text{minimize} & \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) - x_1 - x_2 - x_3 \\ & \text{subject to} & x_1 + x_2 = 1, \\ & & x_2 + x_3 = 3. \end{array}$$

(a) Solve QP1 by using a null-space method. .... (4p)

(b) Solve QP1 by using a Lagrange method. .... (3p)

(c) Use the KKT conditions to prove that the optimal solution to QP1 which you obtained above is an optimal solution also to the inequality-constrained problem QP2 obtained by replacing the constraints $x_1 + x_2 = 1$ and $x_2 + x_3 = 3$ in QP1 by the constraints $x_1 + x_2 \leq 1$ and $x_2 + x_3 \geq 3$ in QP2. .... (3p)

(d) What is the optimal solution to the inequality-constrained problem QP3 obtained by replacing the constraints $x_1 + x_2 = 1$ and $x_2 + x_3 = 3$ in QP1 by the constraints $x_1 + x_2 \geq 1$ and $x_2 + x_3 \leq 3$ in QP3?
Motivate your answer carefully. .... (2p)

4. It is assumed, based on physics, that the quantity w depends on the time t according to the formula

$$w(t) = \frac{1}{1 + c\,t}$$

for some positive (but otherwise unknown) constant c. In order to estimate a value of c one could measure w at different times t, and then calculate the value of c which minimizes the quadratic sum

$$\tfrac{1}{2} \sum_{i=1}^{m} \Big(\frac{1}{1 + c\,t_i} - w_i \Big)^2,$$

where $w_1, \ldots, w_m$ are the obtained measured values of w at the given times $t_1, \ldots, t_m$. Assume the following data: $m = 2$, $t_1 = 1$, $t_2 = 3$, $w_1 = 0.46$, $w_2 = 0.22$.

Start with the guess $c = 1$ and make one iteration with Gauss-Newtons method to calculate a better value of c in the above meaning. .... (7p)

5. Let $\mathbf{D}$ be a given $n \times n$ diagonal matrix with strictly positive diagonal elements $d_1, \ldots, d_n$, where $d_j > 0$ for all j, and let the set $\mathcal{F}$ be defined by

$$\mathcal{F} = \{\mathbf{x} \in I\!R^n \mid \mathbf{x}^\mathsf{T}\mathbf{D}\mathbf{x} \leq 1\}.$$

($\mathcal{F}$ is the region inside and on the surface of an "ellipsoid" in $I\!R^n$.)

Further, let $\mathbf{q} \in I\!R^n$ be a given point outside $\mathcal{F}$, so that $\mathbf{q}^\mathsf{T}\mathbf{D}\mathbf{q} > 1$.

Assume that we want to find the point $\hat{\mathbf{x}}$ in $\mathcal{F}$ with the smallest distance to $\mathbf{q}$ among all the points in $\mathcal{F}$. This can be formulated as a nonlinear optimization problem with a quadratic objective function and a quadratic constraint function:

$$\text{P:} \quad \text{minimize } (\mathbf{x}-\mathbf{q})^\mathsf{T}(\mathbf{x}-\mathbf{q}) \text{ subject to } \mathbf{x}^\mathsf{T}\mathbf{D}\mathbf{x} \leq 1.$$

(a) Use Lagrange relaxation to deduce an *explicit* expression for the dual objective function $\varphi(y)$, valid for all $y \geq 0$, and formulate the dual problem D. .... (4p)

(b) Deduce an explicit expression for the derivative $\varphi'(y)$ of the dual objective function, valid for all $y \geq 0$.
Then show that $\varphi'(0) > 0$, and that $\varphi'(y)$ is strictly decreasing for all $y \geq 0$, and that there is a number $y_1 > 0$ such that $\varphi'(y_1) < 0$. .... (3p)

(c) From (b) above it follows that there is a unique optimal solution $\hat{y}$ to the dual problem. It is in general not possible to calculate $\hat{y}$ analytically, but assume that $\hat{y}$ has been calculated by some numerical method. Then let $\hat{\mathbf{x}}$ be the unique $\mathbf{x}$ which minimizes the Lagrange function $L(\mathbf{x}, \hat{y})$, for fixed $y = \hat{y}$.
Prove that $\hat{\mathbf{x}}$ is an optimal solution to the primal problem P. .... (4p)

Good luck!