KTH Matematik

Exam in SF1811/SF1841 Optimization.
Tuesday, June 3, 2014, 8:00–13:00

Examiner: Krister Svanberg, telephone: 790 7137, email: krille@math.kth.se.

Allowed utensils: Pen, paper, eraser and ruler. (Penna, papper, suddgummi och linjal.)
No calculator! (Ingen räknare!) A formula-sheet is handed out.

Language: Your solutions should be written in English or in Swedish.

Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.
A passing grade E is guarranteed for 25 points, including bonus points from the home assignments during Nov-Dec 2013. 23-24 points give a possibility to complement the exam to grade E within three weeks from the announcement of the results. Contact the examiner as soon as possible by email if this is the case.
Write your name on each page of the solutions you hand in and number the pages.
Write the solutions to the different questions 1,2,3,4,5 on separate sheets.
(This is important since the exams are split up during the corrections.)

---

1. (a) A certain company has two factories, called F1 and F2, and four customers, called C1, C2, C3 and C4.

At a certain occasion, the customers demand the following quantities of the company's product: C1: 70 units, C2: 60 units, C3: 50 units, C4: 40 units.

At the same occasion, the factories have the following supplies of the company's product: F1: 105 units, F2: 115 units.

The transportation costs per units from factories to customers are given by the following table:

	C1	C2	C3	C4
F1	7	6	8	6
F2	6	4	5	3

The company should decide what quantities to transport from each factory to each customer. This should be done in such a way that the total transportation cost is minimized subject to the constraints that each customer receives its demand and each factory uses its current supply. (This is possible to do since the total demand = the total supply = 220 units.)

Show that the following transportation plan gives an optimal solution:

	C1	C2	C3	C4
F1	70	35		
F2		25	50	40

Hint: The problem can formulated as a minimum cost network flow problem with six nodes. . . . . (5p)

(b) Let $\mathbf{H} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & k \end{bmatrix}$.

Decide for which value on the constant k there is an infinite number of global optimal solutions to the problem of minimizing the function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x}$ without any constraints, where $\mathbf{x} \in I\!R^3$.
Also, for this value of k, describe the set of global optimal solutions $\mathbf{x}$. . . (5p)

2. Consider the following linear programming problem:

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + x_2 \geq 2, \\ & x_1 + x_3 \geq 2, \\ & x_2 + x_3 \geq 2, \\ & x_j \geq 0, \quad j = 1, 2, 3. \end{aligned}$$

(a) Transform the problem to the standard form and solve it using the Simplex method. You must start by letting x_1, x_2 and x_3 be the basic variables (which turns out to give a feasible but not optimal basic solution). . . . . (7p)

(b) Formulate the corresponding dual LP problem and decide an optimal solution to this dual problem. Check that the optimal values of the two problems are equal. . . . . (3p)

The following information may be useful in (a): $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

3. Consider the nonlinear optimization problem of minimizing $f(\mathbf{x})$,
where $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T} \in I\!R^3$ and the function $f : I\!R^3 \to I\!R$ is given by

$$f(\mathbf{x}) = \sum_{j=1}^{3} (x_j^4 + x_j^3 + x_j^2 + x_j)$$

.

(a) Use $\mathbf{x}^{(1)} = (1, 0, -1)^\mathsf{T}$ as the starting point, and calculate the next iteration point $\mathbf{x}^{(2)}$ by Newtons method. . . . . (6p)

(b) Is f a convex function on the whole set $I\!R^3$? Motivate your answer. . . . . . (4p)

(c) Calculate a lower bound of the optimal value of the above minimization problem, i.e. a number L such that $f(\mathbf{x}) \geq L$ for all $\mathbf{x} \in I\!R^3$.
The higher lower bound L you find, the more points you get. . . . . (2p)

4. Let the ellipse E be defined by the equation $\dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_2^2} = 1$,
where the given constants a_1 and a_2 are strictly positive.

(a) The perimeter (omkrets) of a rectangle is the sum of its four sides.
Among all rectangels inside the given ellipse E, you should calculate the one with largest perimeter.
You may assume that the optimal rectangle has its four corners in points (x_1, x_2), $(-x_1, x_2)$, $(x_1, -x_2)$ and $(-x_1, -x_2)$ on the ellipse.
Formulate this problem as a nonlinear optimization problem with a constraint. Also formulate the relevant optimality conditions for the problem, and use these conditions to calculate the width and height of the optimal rectangle. . . . (4p)

(b) Repeat the above exercise, but assume now that the *area* of the rectangle should be maximized instead of the perimeter. . . . . (4p)

5. Consider the following (primal) problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} \frac{c_j}{x_j} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j \le V, \\ & 0.1 \le x_j \le 1, \; j = 1, \ldots, n, \end{aligned}$$

where $c_j > 0$ are given positive numbers and V is a given positive number such that $0.1\,n < V < n$ (so that the above problem has feasible solutions).

(a) Use Lagrange relaxation with respect to the single constraint $\sum_{j=1}^{n} x_j \le V$
(while the remaining constraints are considered as implicit constraints) to deduce *explicit* expressions (formulas) for the dual objective function $\varphi(y)$, valid for $y \ge 0$, and formulate the dual problem D. . . . . (5p)

(b) Now assume that $n = 2$, $V = 1.5$, $c_1 = 1$ and $c_2 = 9$.
Deduce explicit expressions for the derivative $\varphi'(y)$ of the dual objective function, valid for $y \ge 0$.
Then calculate an optimal solution $\hat{y}$ to the dual problem and an optimal solution $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)^\mathsf{T}$ to the primal problem.
Also check that the optimal values of the primal and dual problems are equal in this example. . . . . (5p)

Good luck!