KTH Matematik

Exam in SF1811 Optimization.
March 14, 2016, 8:00–13:00.

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Allowed utensils: Pen, paper, eraser and ruler. (Penna, papper, suddgummi och linjal.)
No calculator! (Ingen räknare!) A formula-sheet is handed out.

Language: Your solutions should be written in English or in Swedish.

Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.
A passing grade E is guarranteed for 25 points, including bonus points from the home assignments during Nov-Dec 2015. 23-24 points give a possibility to complement the exam to grade E within three weeks from the announcement of the results. Contact the examiner as soon as possible by email if this is the case.
Write your name on each page of the solutions you hand in and number the pages.
Write the solutions to the different exercises 1,2,3,4,5 on separate sheets.
This is important since the exams are split up during the corrections.

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1. Consider the following LP problem in the variable vector $\mathbf{x} \in I\!R^6$:

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{pmatrix} 700 \\ 800 \\ -400 \\ -500 \\ -600 \end{pmatrix}, \quad \mathbf{c} = (5, 3, 7, K, 2, 4)^{\mathsf{T}}.$$

The fourth component K in the vector $\mathbf{c}$ is a given real number.

(a) As can be seen from the structure of $\mathbf{A}$, this is in fact a minimum cost network flow problem. Illustrate the corresponding network in a figure. . (1p)

(b) Assume that $K = 5$. Show that $\tilde{\mathbf{x}} = (400,\ 300,\ 0,\ 0,\ 200,\ 600)^{\mathsf{T}}$ is an optimal solution to the problem. . . . . (3p)

(c) Assume that $K = 3$. Use the simplex method for network flow problems to solve the problem. Start from the solution in (b). . . . . (4p)

(d) Draw a graph which illustrates how the optimal value of the above problem depends on the value of K in the interval $2 \leq K \leq 6$.
Use a coordinate system with $K \in [\,2,\ 6\,]$ on the horizontal axis and the optimal value of the objective function on the vertical axis. . . . . (2p)

2. Let the quadratic function $f : I\!R^3 \to I\!R$ be defined by:

$$f(\mathbf{x}) = -x_1x_2 - x_2x_3 - x_3x_1.$$

(a) First use a *nullspace method* to minimize $f(\mathbf{x})$ subject to the single constraint

$$x_1 + 2x_2 + 3x_3 = 4.$$

Then check that the Lagrange conditions are satisfied by your obtained solution. What is the value of the Lagrange multiplier for the single constraint? .. (6p)

(b) Assume now that the above constraint is changed to $x_1 - 2x_2 + 3x_3 = 4$. Show that there is no optimal solution to the problem of minimizing $f(\mathbf{x})$ subject to this new single constraint.
Moreover, calculate two vectors $\bar{\mathbf{x}} \in I\!R^3$ and $\mathbf{d} \in I\!R^3$ such that the vector $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^\mathsf{T}$, defined by $\mathbf{x}(t) = \bar{\mathbf{x}} + t \cdot \mathbf{d}$, satisfies $x_1(t) - 2x_2(t) + 3x_3(t) = 4$ for all $t \in I\!R$ and $f(\mathbf{x}(t)) \to -\infty$ when $t \to \infty$. (4p)

3. Consider the following LP problem on standard form:

$$\begin{array}{lll} \text{P:} & \text{minimize} & -3x_1 - 4x_2 - 2x_3 \\ & \text{subject to} & x_1 + 2x_2 + 2x_3 + x_4 \qquad = 180, \\ & & 2x_1 + 2x_2 + x_3 \qquad + x_5 = 120, \\ & & x_j \geq 0,\ j = 1, \ldots, 5. \end{array}$$

(a) Use the simplex method to calculate an optimal solution to this problem. Start with x_4 and x_5 as basic variables. .... (5p)

(b) Formulate the dual LP problem D corresponding to the above problem P. Illustrate this dual problem in a careful figure, and calculate graphically an optimal dual solution. .... (2p)

(c) Formulate the complementarity conditions for the above pair of primal and dual LP problems. Use these optimality conditions, and your results above, to calculate the complete set of optimal solutions to P. .... (3p)

4. Let the nonlinear function $f : I\!R^2 \to I\!R$ be defined by

$$f(\mathbf{x}) = x_1^3 + x_2^3 - 3x_1x_2, \quad \text{where } \mathbf{x} = (x_1, x_2)^\mathsf{T}.$$

In (a) and (b), we consider the problem of minimizing $f(\mathbf{x})$ without any constraints.

(a) Perform a complete iterations with *Newtons method*, starting from the point $\mathbf{x}^{(1)} = (2,\, 2)^\mathsf{T}$. .... (5p)

(b) Calculate (analytically) all the local optimal solutions to the problem. .. (2p)

(c) Let q be a given real number and consider the problem:
P1: minimize $f(\mathbf{x})$ subject to $x_1 + x_2 + q = 0$.
Calculate all points which satisfy the Lagrange optimality conditions for P1. Do this first for $q = -1$, then for $q = 1$. What can you say about local and/or global optimality of your obtained points? .... (3p)

5. Let a_1, a_2 and b be three given real numbers which satisfy

$$a_1 > 0,\ \ a_2 > 0,\ \ a_1^2 + a_2^2 = 1 \ \text{ and } \ b > 0,$$

and let the functions $d_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2, 3$, be defined by

$$d_1(\mathbf{x}) = x_1, \quad d_2(\mathbf{x}) = x_2 \ \text{ and } \ d_3(\mathbf{x}) = b - a_1 x_1 - a_2 x_2.$$

Then the lines L_1, L_2 and L_3, defined by $L_i = \{\, \mathbf{x} \in \mathbb{R}^2 \mid d_i(\mathbf{x}) = 0 \,\}$, for $i = 1, 2, 3$, (i.e. the three lines "$x_1 = 0$", "$x_2 = 0$" and "$a_1 x_1 + a_2 x_2 = b$") generate a triangle with corner points $(0\,,\,0)^{\mathsf{T}}$, $(b/a_1\,,\,0)^{\mathsf{T}}$ and $(0\,,\,b/a_2)^{\mathsf{T}}$. The set T of points in this triangle (including points at the boundary of the triangle) can be expressed as

$$T = \{\, \mathbf{x} \in \mathbb{R}^2 \mid d_i(\mathbf{x}) \geq 0,\ i = 1, 2, 3 \,\},$$

and we have the following interpretation of the functions d_i:

If $\mathbf{x} \in T$ then $d_i(\mathbf{x})$ is the *distance* from $\mathbf{x}$ to L_i, for $i = 1, 2, 3$.

This is obvious for $i = 1$ and $i = 2$. For $i = 3$, it follows from an elementary result in linear algebra and geometry, using that $a_1^2 + a_2^2 = 1$. (You do not need to show that.)

Note: In the following exercises, you are not forced to use "systematic" search methods in every part, but you must of course always prove optimality of your suggested solutions. As an example, a permitted method could be to illustrate a problem graphically and make a "qualified guess" of the optimal solution $\hat{\mathbf{x}}$, and then prove that $\hat{\mathbf{x}}$ is optimal.

(a) First, consider the problem of minimizing the *sum* of the above distances:

$$\mathrm{P}_1: \quad \text{minimize } \ d_1(\mathbf{x}) + d_2(\mathbf{x}) + d_3(\mathbf{x}) \ \text{ subject to } \ \mathbf{x} \in T.$$

Calculate an optimal solution to P_1. .... (3p)

(b) Next, consider the problem of minimizing the *sum of squares* of the above distances:

$$\mathrm{P}_2: \quad \text{minimize } \ (d_1(\mathbf{x}))^2 + (d_2(\mathbf{x}))^2 + (d_3(\mathbf{x}))^2 \ \text{ subject to } \ \mathbf{x} \in T.$$

Calculate an optimal solution to P_2. .... (3p)

Hint: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $ad \neq cb$.

(c) Finally, consider the problem of minimizing the *largest* of the above distances.

$$\mathrm{P}_3: \quad \text{minimize } \ \max\{d_1(\mathbf{x}), d_2(\mathbf{x}), d_3(\mathbf{x})\} \ \text{ subject to } \ \mathbf{x} \in T,$$

where $\max\{\alpha, \beta, \gamma\}$ denotes the largest of the three numbers α, β and γ. Calculate an optimal solution to P_3.
Hint: To verify optimality of a suggested solution, you may reformulate P_3 to an equivalent LP problem, and then use the optimality conditions. ... (4p)

Good luck!