Formula sheet on the exam in SF1811, Jan 2016

Note: No calculator is allowed on the exam!

$\mathcal{R}(\mathbf{A})^\perp = \mathcal{N}(\mathbf{A}^\mathsf{T})$, $\mathcal{R}(\mathbf{A}^\mathsf{T})^\perp = \mathcal{N}(\mathbf{A})$, $\mathcal{N}(\mathbf{A})^\perp = \mathcal{R}(\mathbf{A}^\mathsf{T})$, $\mathcal{N}(\mathbf{A}^\mathsf{T})^\perp = \mathcal{R}(\mathbf{A})$.

Simplex method for LP problem on standard form.
$\mathbf{A}_\beta = [\,\mathbf{a}_{\beta_1} \cdots \mathbf{a}_{\beta_m}]$, $\mathbf{A}_\nu = [\,\mathbf{a}_{\nu_1} \cdots \mathbf{a}_{\nu_\ell}]$, $\mathbf{A}_\beta \bar{\mathbf{b}} = \mathbf{b}$, $\bar{z} = \mathbf{c}_\beta^\mathsf{T} \bar{\mathbf{b}}$, $\mathbf{A}_\beta^\mathsf{T} \mathbf{y} = \mathbf{c}_\beta$, $\mathbf{r}_\nu^\mathsf{T} = \mathbf{c}_\nu^\mathsf{T} - \mathbf{y}^\mathsf{T} \mathbf{A}_\nu$.
Finished if $\mathbf{r}_\nu \geq \mathbf{0}$. Otherwise, choose a q with $r_{\nu_q} < 0$. $k = \nu_q$, $\mathbf{A}_\beta \bar{\mathbf{a}}_k = \mathbf{a}_k$, $x_k = t$,
$z = \bar{z} + r_k t$, $\mathbf{x}_\beta = \bar{\mathbf{b}} - \bar{\mathbf{a}}_k t$, $t^{\max} = \min\limits_i \left\{ \dfrac{\bar{b}_i}{\bar{a}_{ik}} \mid \bar{a}_{ik} > 0 \right\} = \dfrac{\bar{b}_p}{\bar{a}_{pk}}$. Let ν_q och β_p change place.

$$\begin{array}{lll} \text{P:} & \text{minimize} & \mathbf{c}_1^\mathsf{T}\mathbf{x}_1 + \mathbf{c}_2^\mathsf{T}\mathbf{x}_2 \\ & \text{subject to} & \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2 \geq \mathbf{b}_1, \\ & & \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{x}_2 = \mathbf{b}_2, \\ & & \mathbf{x}_1 \geq \mathbf{0}, \quad \mathbf{x}_2 \text{ free.} \end{array} \qquad \begin{array}{lll} \text{D:} & \text{maximize} & \mathbf{b}_1^\mathsf{T}\mathbf{y}_1 + \mathbf{b}_2^\mathsf{T}\mathbf{y}_2 \\ & \text{subject to} & \mathbf{A}_{11}^\mathsf{T}\mathbf{y}_1 + \mathbf{A}_{21}^\mathsf{T}\mathbf{y}_2 \leq \mathbf{c}_1, \\ & & \mathbf{A}_{12}^\mathsf{T}\mathbf{y}_1 + \mathbf{A}_{22}^\mathsf{T}\mathbf{y}_2 = \mathbf{c}_2, \\ & & \mathbf{y}_1 \geq \mathbf{0}, \quad \mathbf{y}_2 \text{ free.} \end{array}$$

$\hat{\mathbf{x}}$ is optimal to P and $\hat{\mathbf{y}}$ is optimal D *if and only if* $\hat{\mathbf{x}}$ is feasible to P, $\hat{\mathbf{y}}$ is feasible to D,
$\hat{\mathbf{y}}_1^\mathsf{T}(\mathbf{A}_{11}\hat{\mathbf{x}}_1 + \mathbf{A}_{12}\hat{\mathbf{x}}_2 - \mathbf{b}_1) = 0$ and $\hat{\mathbf{x}}_1^\mathsf{T}(\mathbf{c}_1 - \mathbf{A}_{11}^\mathsf{T}\hat{\mathbf{y}}_1 - \mathbf{A}_{21}^\mathsf{T}\hat{\mathbf{y}}_2) = 0$.

A symmetric matrix $\mathbf{H}$ is positive definite [semidefinite] if and only if there is a lower triangular matrix $\mathbf{L}$ with all $\ell_{ii} = 1$ and a diagonal matrix $\mathbf{D}$ with all $d_i > 0$ [$d_i \geq 0$] such that $\mathbf{H} = \mathbf{L}\mathbf{D}\mathbf{L}^\mathsf{T}$.

Quadratic functions. $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} + \mathbf{c}^\mathsf{T}\mathbf{x} + c_0$, with $\mathbf{H}$ symmetric.
$\nabla f(\mathbf{x}) = (\mathbf{H}\mathbf{x} + \mathbf{c})^\mathsf{T}$, $\mathbf{F}(\mathbf{x}) = \mathbf{H}$, $f(\mathbf{x} + t\,\mathbf{d}) = f(\mathbf{x}) + t\,(\mathbf{H}\mathbf{x} + \mathbf{c})^\mathsf{T}\mathbf{d} + \frac{1}{2}\,t^2\,\mathbf{d}^\mathsf{T}\mathbf{H}\mathbf{d}$.
$\hat{\mathbf{x}}$ minimizes $f(\mathbf{x})$ if and only if $\mathbf{H}$ is positive semidefinite and $\mathbf{H}\hat{\mathbf{x}} + \mathbf{c} = \mathbf{0}$.

Equality-constrained QP. minimize $\frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} + \mathbf{c}^\mathsf{T}\mathbf{x} + c_0$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$.
$\hat{\mathbf{x}} = \bar{\mathbf{x}} + \mathbf{Z}\hat{\mathbf{v}}$, $(\mathbf{Z}^\mathsf{T}\mathbf{H}\mathbf{Z})\hat{\mathbf{v}} = -\mathbf{Z}^\mathsf{T}(\mathbf{H}\bar{\mathbf{x}} + \mathbf{c})$, $\begin{bmatrix} \mathbf{H} & -\mathbf{A}^\mathsf{T} \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{u}} \end{pmatrix} = \begin{pmatrix} -\mathbf{c} \\ \mathbf{b} \end{pmatrix}$.

MN solution to LSQ problems. $\mathbf{A}^\mathsf{T}\mathbf{A}\bar{\mathbf{x}} = \mathbf{A}^\mathsf{T}\mathbf{b}$, $\mathbf{A}\mathbf{A}^\mathsf{T}\bar{\mathbf{u}} = \mathbf{A}\bar{\mathbf{x}}$, $\hat{\mathbf{x}} = \mathbf{A}^\mathsf{T}\bar{\mathbf{u}}$.

Newton. $\mathbf{F}(\mathbf{x}^{(k)})\mathbf{d} = -\nabla f(\mathbf{x}^{(k)})^\mathsf{T}$ gives $\mathbf{d}^{(k)}$ if $\mathbf{F}(\mathbf{x}^{(k)})$ is positive definite.
$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + t_k\mathbf{d}^{(k)}$ where t_k satisfies $f(\mathbf{x}^{(k)} + t_k\mathbf{d}^{(k)}) < f(\mathbf{x}^{(k)})$. Try $t_k = 1$ first.

Nonlinear LSQ. minimize $f(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^m (h_i(\mathbf{x}))^2 = \frac{1}{2}\mathbf{h}(\mathbf{x})^\mathsf{T}\mathbf{h}(\mathbf{x})$.
$\mathbf{x} \in I\!R^n$, $\mathbf{h}(\mathbf{x}) \in I\!R^m$, $\nabla\mathbf{h}(\mathbf{x})$ a $m \times n$ matrix with $\dfrac{\partial h_i}{\partial x_j}(\mathbf{x})$ in row i and column j.
$\nabla f(\mathbf{x}) = \mathbf{h}(\mathbf{x})^\mathsf{T}\nabla\mathbf{h}(\mathbf{x})$ (row vector), $\mathbf{F}(\mathbf{x}) = \nabla\mathbf{h}(\mathbf{x})^\mathsf{T}\nabla\mathbf{h}(\mathbf{x}) + \sum_i h_i(\mathbf{x})\mathbf{H}_i(\mathbf{x})$.

Gauss-Newton. $\nabla\mathbf{h}(\mathbf{x}^{(k)})^\mathsf{T}\nabla\mathbf{h}(\mathbf{x}^{(k)})\mathbf{d} = -\nabla\mathbf{h}(\mathbf{x}^{(k)})^\mathsf{T}\mathbf{h}(\mathbf{x}^{(k)})$ gives $\mathbf{d}^{(k)}$.
$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + t_k\mathbf{d}^{(k)}$ where t_k satisfies $f(\mathbf{x}^{(k)} + t_k\mathbf{d}^{(k)}) < f(\mathbf{x}^{(k)})$. Try $t_k = 1$ first.

Equality-constrained NLP. minimize $f(\mathbf{x})$ subject to $h_i(\mathbf{x}) = 0$, $i = 1, \ldots, m$.
Lagrange conditions: $\nabla f(\hat{\mathbf{x}}) + \sum_i \hat{u}_i \nabla h_i(\hat{\mathbf{x}}) = \mathbf{0}^\mathsf{T}$ and $h_i(\hat{\mathbf{x}}) = 0$, $i = 1, \ldots, m$.

Inequality-constrained NLP. minimize $f(\mathbf{x})$ subject to $g_i(\mathbf{x}) \leq 0$, $i = 1, \ldots, m$.
KKT conditions: $\nabla f(\hat{\mathbf{x}}) + \sum_i \hat{y}_i \nabla g_i(\hat{\mathbf{x}}) = \mathbf{0}^\mathsf{T}$, $g_i(\hat{\mathbf{x}}) \leq 0$, $\hat{y}_i \geq 0$, $\hat{y}_i g_i(\hat{\mathbf{x}}) = 0$.

Lagrangean relaxation. P: minimize $f(\mathbf{x})$ subject to $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ and $\mathbf{x} \in X$.
$L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \mathbf{y}^\mathsf{T}\mathbf{g}(\mathbf{x})$, $\varphi(\mathbf{y}) = \min\limits_{\mathbf{x} \in X} L(\mathbf{x}, \mathbf{y})$. D: maximize $\varphi(\mathbf{y})$ s.t. $\mathbf{y} \geq \mathbf{0}$.
Glob. opt. cond. (GOC): $L(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \min\limits_{\mathbf{x} \in X} L(\mathbf{x}, \hat{\mathbf{y}})$, $\mathbf{g}(\hat{\mathbf{x}}) \leq \mathbf{0}$, $\hat{\mathbf{y}} \geq \mathbf{0}$, $\hat{\mathbf{y}}^\mathsf{T}\mathbf{g}(\hat{\mathbf{x}}) = 0$.