Solutions to exam in SF1811 Optimization, April 7, 2015

1.(a)+(b)

The network corresponding to the given LP problem can be illustrated by the left figure below, where the supply at the nodes (i.e. the components in the vector $\mathbf{b}$), and the unit costs of the arcs (i.e. the components in the vector $\mathbf{c}$) are written in the figure.
All arcs are directed from left to right. Negative supply means demand.
The suggested solution $\hat{\mathbf{x}} = (4, 0, 9, 0, 15, 0, 8)^\mathsf{T}$ can be illustrated by the spanning tree in the right figure below, with the arc-flows written on the arcs.

```
      Nod 2      Nod 4
       +5          -7                      +5          -7
        0 - - - - - 0                       0           0
       / \    4    / \                     / \         / \
      /   \       /   \                   /   \       /   \
    5/    1\     /2    \3               4/    9\     /15   \8
    /       \   /       \               /       \   /       \
   /    7    \ /    6    \             /         \ /         \
  0 - - - - - 0 - - - - - 0           0           0           0
 +4          +6          -8          +4          +6          -8
Nod 1       Nod 3       Nod 5
```

The simplex multipliers y_i for the nodes are calculated from $y_5 = 0$ and $y_i - y_j = c_{ij}$ for all ars (i, j) in the spanning tree (left figure below), whereafter the reduced cost for the non-basic variables are calculated from $r_{ij} = c_{ij} - y_i + y_j$ (right figure below).

```
      y2=6        y4=3                     y2=6    c24=4    y4=3
        0           0                        0 - - - - - 0
       / \         / \                         r24=4-6+3=1
      /   \       /   \
    5/    1\     /2    \3
    /       \   /       \
   /         \ /         \                r13=7-11+5=1   r35=6-5+0=1
  0           0           0               0 - - - - - 0 - - - - - 0
y1=11       y3=5        y5=0          y1=11   c13=7   y3=5   c35=6   y5=0
```

Since all $r_{ij} \geq 0$, the suggested solution $\hat{\mathbf{x}}$ is optimal.

1.(c) When the primal problem is on the standard form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

the corresponding dual problem becomes maximize $\mathbf{b}^\mathsf{T}\mathbf{y}$ subject to $\mathbf{A}^\mathsf{T}\mathbf{y} \leq \mathbf{c}$, which here becomes

$$\begin{aligned} \text{maximize} \quad & 4y_1 + 5y_2 + 6y_3 - 7y_4 - 8y_5 \\ \text{subject to} \quad & y_1 - y_2 \leq 5, \\ & y_1 - y_3 \leq 7, \\ & y_2 - y_3 \leq 1, \\ & y_2 - y_4 \leq 4, \\ & y_3 - y_4 \leq 2, \\ & y_3 - y_5 \leq 6, \\ & y_4 - y_5 \leq 3. \end{aligned}$$

It is well known that an optimal solution to this dual problem is given by the vector $\mathbf{y}$ with simplex multipliers from 1.(b), i.e. $\mathbf{y} = (11, 6, 5, 3, 0)^\mathsf{T}$.
Then the right hand sides minus the left hand sides of the dual constraint become $\mathbf{c} - \mathbf{A}^\mathsf{T}\mathbf{y} = (0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0)^\mathsf{T} \geq \mathbf{0}^\mathsf{T}$, so $\mathbf{y}$ is a feasible solution to the dual problem.

Moreover, since $\hat{\mathbf{x}} = (4, 0, 9, 0, 15, 0, 8)^\mathsf{T}$, we have that $\hat{\mathbf{x}}^\mathsf{T}(\mathbf{c} - \mathbf{A}^\mathsf{T}\mathbf{y}) = 0$, which shows that the complementarity conditions are satisfied. Thus $\mathbf{y}$ is optimal to the dual problem, and $\hat{\mathbf{x}}$ is optimal to the primal problem (which we already knew).

1.(d) If the right hand side vector $\mathbf{b}$ is changed from $(4,\ 5,\ 6, -7, -8)^\mathsf{T}$ to $(7,\ 8, -4, -5, -6)^\mathsf{T}$, the arc-flows corresponding to the spanning tree from 1.(b) become

```
     +8         -5
      0          0
     / \        / \
    /   \      /   \
  7/   15\    /11   \6
  /       \  /       \
 /         \/         \
0           0          0
+7         -4         -6
```

Since all the arc-flows x_{ij} become non-negative, this new solution $\mathbf{x} = (7, 0, 15, 0, 11, 0, 6)^\mathsf{T}$ is a feasible basic solution. Moreover, since the cost-vector $\mathbf{c}$ is the same as in 1.(b), the simplex multipliers y_i and the reduced costs r_{ij} become exactly the same as in 1.(b), i.e $r_{ij} \geq 0$. Therefore, this new solution $\mathbf{x}$ is an optimal solution to the new problem.

The corresponding dual problem looks the same as in 1.(c), except for the objective function which is now $7y_1 + 8y_2 - 4y_3 - 5y_4 - 6y_5$.
With $\mathbf{b} = (7,\ 8, -4, -5, -6)^\mathsf{T}$ and $\mathbf{y} = (11, 6, 5, 3, 0)^\mathsf{T}$, the dual objective value now becomes $\mathbf{b}^\mathsf{T}\mathbf{y} = 77 + 48 - 20 - 15 - 0 = 90$,
while the primal objective value, with $\mathbf{c}^\mathsf{T} = (5,\ 7,\ 1,\ 4,\ 2,\ 6,\ 3)$ and $\mathbf{x} = (7, 0, 15, 0, 11, 0, 6)^\mathsf{T}$, becomes $\mathbf{c}^\mathsf{T}\mathbf{x} = 35 + 15 + 22 + 18 = 90$. Thus, $\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{b}^\mathsf{T}\mathbf{y}$, as it should be.

2.

The search for the optimal points $\mathbf{y} \in L_1$ and $\mathbf{z} \in L_2$ can be formulated as the QP problem

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}\| \mathbf{y} - \mathbf{z} \|^2 = \tfrac{1}{2}(\mathbf{y} - \mathbf{z})^{\mathsf{T}}(\mathbf{y} - \mathbf{z}) \\ \text{subject to} \quad & \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ & \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \mathbf{z} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

As the hint indicates, a nullspace method may make sense:

The system $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, with the general solution $\mathbf{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \mathbf{y}_0 + t\cdot\mathbf{d}$, where t is an arbitrary real number.

The system $\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \mathbf{z} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is equivalent to $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{z} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, with the general solution $\mathbf{z} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + s\cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \mathbf{z}_0 + s\cdot\mathbf{p}$, where s is an arbitrary real number.

If these expressions are plugged into the objective function, the following unconstrained QP problem in the variables t and s is obtained: minimize $\tfrac{1}{2}\| \mathbf{y}_0 + t\cdot\mathbf{d} - \mathbf{z}_0 - s\cdot\mathbf{p} \|^2$.

One of several methods for solving this problem is as the least squares problem

minimize $\tfrac{1}{2}\| \mathbf{A}\mathbf{x} - \mathbf{b} \|^2$, where $\mathbf{x} = \begin{pmatrix} t \\ s \end{pmatrix}$, $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{b} = \mathbf{z}_0 - \mathbf{y}_0 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$.

The normal equations $\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$ become $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{t} \\ \hat{s} \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$.

Then $\hat{\mathbf{y}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \hat{t}\cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/3 \\ -1/3 \end{pmatrix}$ and $\hat{\mathbf{z}} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \hat{s}\cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ -1 \end{pmatrix}$

are the optimal points we searched for.

Thus, the shortest distance between the lines is $\| \hat{\mathbf{y}} - \hat{\mathbf{z}} \| = \dfrac{2}{\sqrt{3}}$.

3.

The objective function is $f(\mathbf{x}) = x_1^3 - 3x_1 + x_1x_2 + \frac{1}{2}x_1^2x_2^2$,

with gradient $\nabla f(\mathbf{x})^\mathsf{T} = \begin{pmatrix} 3x_1^2 - 3 + x_2 + x_1x_2^2 \\ x_1 + x_1^2x_2, \end{pmatrix}$,

and Hessian $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 6x_1 + x_2^2 & 1 + 2x_1x_2 \\ 1 + 2x_1x_2 & x_1^2 \end{bmatrix}$.

We will use the well known fact that a symmetric 2×2 matrix $\mathbf{H} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$
* is positive definite if and only if $a > 0$, $c > 0$ and $ac - b^2 > 0$,
* is positive semidefinite if and only if $a \geq 0$, $c \geq 0$ and $ac - b^2 \geq 0$,
which is easily verified, e.g. by an LDLT factorization.

3.(a) If $\mathbf{x} = (0, 3)^\mathsf{T}$ then $\nabla f(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 9 & 1 \\ 1 & 0 \end{bmatrix}$, which is not positive semidefinite. Thus, $\mathbf{x} = (0, 3)^\mathsf{T}$ is *not* a local minimum point.

If $\mathbf{x} = (1, -1)^\mathsf{T}$ then $\nabla f(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 7 & -1 \\ -1 & 1 \end{bmatrix}$, which is positive definite. Thus, $\mathbf{x} = (1, -1)^\mathsf{T}$ *is* a local minimum point.

If $\mathbf{x} = (-1, 1)^\mathsf{T}$ then $\nabla f(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} -5 & -1 \\ -1 & 1 \end{bmatrix}$, which is not positive semidefinite. Thus, $\mathbf{x} = (-1, 1)^\mathsf{T}$ is *not* a local minimum point.

3.(b) The given starting point for Newtons method is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, with $f(\mathbf{x}^{(1)}) = -2$.

Then $\nabla f(\mathbf{x}^{(1)})^\mathsf{T} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 6 & 1 \\ 1 & 1 \end{bmatrix}$, which is positive definite.

The Newton direction $\mathbf{d}^{(1)}$ is then obtained as the solution to $\mathbf{F}(\mathbf{x}^{(1)})\mathbf{d} = -\nabla f(\mathbf{x}^{(1)})^\mathsf{T}$,

which becomes $\begin{bmatrix} 6 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{d} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, with the unique solution $\mathbf{d}^{(1)} = \begin{pmatrix} 0.2 \\ -1.2 \end{pmatrix}$.

We first try the step parameter $t_1 = 1$, so that $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + t_1\mathbf{d}^{(1)} = \mathbf{x}^{(1)} + \mathbf{d}^{(1)} = \begin{pmatrix} 1.2 \\ -1.2 \end{pmatrix}$.

Then $f(\mathbf{x}^{(2)}) = 1.2^3 - 3 \cdot 1.2 - 1.2^2 + 0.6 \cdot 1.2^3 = 1.2^2 \cdot (1.2 - 2.5 - 1 + 0.72) =$
$= -1.44 \cdot 1.58 = -2.2752 < -2 = f(\mathbf{x}^{(1)})$, so the step is accepted.

The Newton iteration is thus completed and we have obtained the new iteration point

$\mathbf{x}^{(2)} = \begin{pmatrix} 1.2 \\ -1.2 \end{pmatrix}$ with $f(\mathbf{x}^{(2)}) = -2.2752$.

3.(c)

First note that for all $\mathbf{x}$ with both $x_1 \geq 0$ and $x_2 \geq 0$ the following holds:

$f(\mathbf{x}) = x_1^3 - 3x_1 + x_1x_2 + \frac{1}{2}x_1^2x_2^2 \;\geq\; x_1^3 - 3x_1.$

Then consider the one-variable function $g(x_1) = x_1^3 - 3x_1$,
with $g'(x_1) = 3x_1^2 - 3$ and $g''(x_1) = 6x_1$.

Since $g''(x_1) \geq 0$ for all $x_1 \geq 0$, $g(x_1)$ is a *convex* function on the convex set $\{x_1 \in I\!R \mid x_1 \geq 0\}$.

But since $g'(1) = 0$, it then follows that $x_1 = 1$ is global minimum point of the convex function $g(x_1)$ on the convex set $\{x_1 \in I\!R \mid x_1 \geq 0\}$,
which means that $x_1^3 - 3x_1 \;\geq g(1) = -2$ for all $x_1 \geq 0$.

By combining the above observations, we get that the following inequalities hold for all $\mathbf{x}$ with $x_1 \geq 0$ and $x_2 \geq 0$:

$f(\mathbf{x}) = x_1^3 - 3x_1 + x_1x_2 + \frac{1}{2}x_1^2x_2^2 \;\geq\; x_1^3 - 3x_1 \;\geq -2.$

But the point $\hat{\mathbf{x}} = (1, 0)^\mathsf{T}$ satisfies $\hat{x}_1 \geq 0$, $\hat{x}_2 \geq 0$ and $f(\hat{\mathbf{x}}) = -2$.

Thus, $\hat{\mathbf{x}} = (1, 0)^\mathsf{T}$ is a global optimal solution to the problem of minimizing $f(\mathbf{x})$ subject to the constraints $x_1 \geq 0$ and $x_2 \geq 0$.

4.(a)

With $\beta = (1, 21)$ and $\nu = (2, 3, \ldots, 19, 20)$ we get that $\mathbf{A}_\beta = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$.

The vector $\bar{\mathbf{b}}$ is obtained from $\mathbf{A}_\beta \bar{\mathbf{b}} = \mathbf{b} = (40,\ 20)^\mathsf{T}$, with the solution $\bar{\mathbf{b}} = (2,\ 1)^\mathsf{T}$, i.e. $x_1 = 2$ and $x_{21} = 1$ in the first feasible basic solution.

Reduced costs for the non-basic variables are given by $\mathbf{r}_\nu^\mathsf{T} = \mathbf{c}_\nu^\mathsf{T} - \mathbf{y}^\mathsf{T}\mathbf{A}_\nu$,
where $\mathbf{y}$ is obtained from $\mathbf{A}_\beta^\mathsf{T}\mathbf{y} = \mathbf{c}_\beta = (10,\ 10)^\mathsf{T}$, with the solution $\mathbf{y} = (0.5,\ 0.5)^\mathsf{T}$.

For each non-basic index j, we then get that
$r_j = c_j - \mathbf{y}^\mathsf{T}\mathbf{a}_j = |\, j - 11\,| - 0.5(21 - j) - 0.5(j - 1) = |\, j - 11\,| - 10.$

The smallest reduced cost is obtained for $j = 11$ so we let $k = 11$.
Then $r_k = -10 < 0$ and the non-basic variable $x_k = x_{11}$ should become a basic variable.

The vector $\bar{\mathbf{a}}_k$ is obtained from $\mathbf{A}_\beta \bar{\mathbf{a}}_k = \mathbf{a}_k = (10,\ 10)^\mathsf{T}$, whith the solution $\bar{\mathbf{a}}_k = (0.5,\ 0.5)^\mathsf{T}$.

Since both $\bar{a}_{1k}$ and $\bar{a}_{2k}$ are > 0, we should compare $\dfrac{\bar{b}_1}{\bar{a}_{1k}} = \dfrac{2}{0.5}$ and $\dfrac{\bar{b}_2}{\bar{a}_{2k}} = \dfrac{1}{0.5}$.

The second ratio is smallest, so $x_{\beta_2} = x_{21}$ should become a non-basic variable.

Now $\beta = (1, 11)$ and $\mathbf{A}_\beta = \begin{bmatrix} 20 & 10 \\ 0 & 10 \end{bmatrix}$.

The vector $\bar{\mathbf{b}}$ is obtained from $\mathbf{A}_\beta \bar{\mathbf{b}} = \mathbf{b} = (40,\ 20)^\mathsf{T}$, with the solution $\bar{\mathbf{b}} = (1,\ 2)^\mathsf{T}$, i.e. $x_1 = 1$ and $x_{11} = 2$ in the current feasible basic solution.

Reduced costs for the non-basic variables are given by $\mathbf{r}_\nu^\mathsf{T} = \mathbf{c}_\nu^\mathsf{T} - \mathbf{y}^\mathsf{T}\mathbf{A}_\nu$,
where $\mathbf{y}$ is obtained from $\mathbf{A}_\beta^\mathsf{T}\mathbf{y} = \mathbf{c}_\beta = (10,\ 0)^\mathsf{T}$, with the solution $\mathbf{y} = (0.5,\ -0.5)^\mathsf{T}$.

For each non-basic index j, we then get that
$r_j = c_j - \mathbf{y}^\mathsf{T}\mathbf{a}_j = |\, j - 11\,| - 0.5(21 - j) + 0.5(j - 1) = |\, j - 11\,| + j - 11 \geq 0.$

Thus, the current feasible basic solution $x_1 = 1$, $x_{11} = 2$ and $x_j = 0$ for $j \notin \{1, 11\}$ is an optimal solution, with the optimal value $= c_1 x_1 + c_{11} x_{11} = 10$.

4.(b): Assume that $\beta = (1,\, q)$ where $q \in \{2, 3, \ldots, 21\}$. Then $\mathbf{A}_\beta = \begin{bmatrix} 20 & 21-q \\ 0 & q-1 \end{bmatrix}$.

The vector $\bar{\mathbf{b}}$ is obtained from $\mathbf{A}_\beta \bar{\mathbf{b}} = \mathbf{b} = (40,\, 20)^\mathsf{T}$, with the solution

$\bar{\mathbf{b}} = \left(\dfrac{3q-23}{q-1},\ \dfrac{20}{q-1} \right)^\mathsf{T}$, i.e. $x_1 = \dfrac{3q-23}{q-1}$ and $x_q = \dfrac{20}{q-1}$ when $\beta = (1,\, q)$.

This is a *feasible* basic solution if and only if $3q - 23 \geq 0$, i.e. if and only if $q \in \{8, 9, \ldots, 21\}$.

Thus, there are 14 different feasible basic solutions with x_1 as one of the basic variables.

When $\beta = (1,\, q)$ and $q \in \{8, 9, \ldots, 21\}$, the objective value for the corresponding feasible basic solution is $c_1 x_1 + c_q x_q = 10x_1 + |\, q - 11\,|\, x_q$

We get two cases:

Case 1: $q \in \{8, 9, 10, 11\}$, for which $|\, q - 11\,| = 11 - q$.

Then $c_1 x_1 + c_q x_q = 10x_1 + (11-q)x_q = \dfrac{10(3q-23)}{q-1} + \dfrac{20(11-q)}{q-1} = \dfrac{10q-10}{q-1} = 10.$

Case 2: $q \in \{12, 13, \ldots, 21\}$, for which $|\, q - 11\,| = q - 11$.

Then $c_1 x_1 + c_q x_q = 10x_1 + (11-q)x_q = \dfrac{10(3q-23)}{q-1} + \dfrac{20(q-11)}{q-1} = \dfrac{50q-450}{q-1} =$

$= 10 + \dfrac{40q-440}{q-1} > 10$, since $q \geq 12$.

Thus, there are 4 different optimal basic solutions with x_1 as one of the basic variables, namely the ones in Case 1 above.

5.(a)

With $\mathbf{y} = (y_1, \ldots, y_m)^\mathsf{T}$, the Lagrange function becomes

$$L(z, \mathbf{x}, \mathbf{y}) = \tfrac{1}{2}z^2 + \sum_{i=1}^{m} y_i(\| \mathbf{x} - \mathbf{p}_i\|^2 - z) =$$

$$= \tfrac{1}{2}z^2 - \sum_{i=1}^{m} y_i z \ + \ \sum_{i=1}^{m} y_i \,(\mathbf{x}^\mathsf{T}\mathbf{x} - 2\,\mathbf{p}_i^\mathsf{T}\mathbf{x} + \mathbf{p}_i^\mathsf{T}\mathbf{p}_i) =$$

$$= \tfrac{1}{2}z^2 - (\mathbf{e}^\mathsf{T}\mathbf{y})z + \mathbf{x}^\mathsf{T}\mathbf{x}\,\mathbf{e}^\mathsf{T}\mathbf{y} - 2\,\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{y} + \mathbf{q}^\mathsf{T}\mathbf{y},$$

where $\mathbf{P}$ is a marix with the columns $\mathbf{p}_1, \ldots, \mathbf{p}_m$,
while $\mathbf{q} = (\|\mathbf{p}_1\|^2, \ldots, \|\mathbf{p}_m\|^2)^\mathsf{T}$ and $\mathbf{e} = (1, \ldots, 1)^\mathsf{T}$.

To get the dual objective function, $L(z, \mathbf{x}, \mathbf{y})$ should be minimized with respect to z and $\mathbf{x}$.

If $\mathbf{y} = \mathbf{0}$ then $L(z, \mathbf{x}, \mathbf{0}) = \frac{1}{2}z^2$, and then minimizing z is $z = 0$
while $\mathbf{x}$ can be anything. The dual objective function then becomes $\varphi(\mathbf{0}) = 0$.

If $\mathbf{y} \neq \mathbf{0}$ (and $\mathbf{y} \geq \mathbf{0}$ of course) then $\mathbf{e}^\mathsf{T}\mathbf{y} > 0$, and the minimizing z is $z(\mathbf{y}) = \mathbf{e}^\mathsf{T}\mathbf{y}$
and the minimizing $\mathbf{x}$ is $\mathbf{x}(\mathbf{y}) = \dfrac{\mathbf{P}\mathbf{y}}{\mathbf{e}^\mathsf{T}\mathbf{y}}$. Then the dual objective function becomes

$$\varphi(\mathbf{y}) = L(z(\mathbf{y}), \mathbf{x}(\mathbf{y}), \mathbf{y}) = -\tfrac{1}{2}(\mathbf{e}^\mathsf{T}\mathbf{y})^2 + \mathbf{q}^\mathsf{T}\mathbf{y} - \frac{\mathbf{y}^\mathsf{T}\mathbf{P}^\mathsf{T}\mathbf{P}\mathbf{y}}{\mathbf{e}^\mathsf{T}\mathbf{y}}\,.$$

5.(b)

If $\mathbf{P} = \mathbf{I}$ ($2{\times}2$), and thus $\mathbf{q} = \mathbf{e} = (1, 1)^\mathsf{T}$, then

$$\varphi(\mathbf{y}) = -\tfrac{1}{2}(y_1 + y_2)^2 + y_1 + y_2 - \frac{y_1^2 + y_2^2}{y_1 + y_2}\,.$$

In particular, the suggested vector $\hat{\mathbf{y}} = (0.25,\ 0.25)^\mathsf{T}$ is a feasible solution to the dual problem, with the dual objective value $\varphi(\hat{\mathbf{y}}) = 0.25$.

Let $\hat{z} = z(\hat{\mathbf{y}}) = \mathbf{e}^\mathsf{T}\hat{\mathbf{y}} = 0.5$ and $\hat{\mathbf{x}} = \mathbf{x}(\hat{\mathbf{y}}) = \dfrac{\mathbf{P}\hat{\mathbf{y}}}{\mathbf{e}^\mathsf{T}\hat{\mathbf{y}}} = (0.5,\ 0.5)^\mathsf{T}$.

Then $\hat{z}$ and $\hat{\mathbf{x}}$ satisfy the constraints in the original (primal) problem P, and is thus a feasible solution to P. The primal objective value of this solution is $\frac{1}{2}\hat{z}^2 = 0.25 = \varphi(\hat{\mathbf{y}})$.

According to a well known theorem, this implies that $\hat{z}$ and $\hat{\mathbf{x}}$ is an optimal solution to P while $\hat{\mathbf{y}}$ is an optimal solution to D.