Solutions to exam in SF1811 Optimization, Jan 13, 2016

1.(a) and **1.(b)**

The network corresponding to the given equations $\mathbf{Ax} = \mathbf{b}$ can be illustrated by the left figure below, where the supply at the nodes, i.e. the components in the vector $\mathbf{b}$, are written in the figure. All arcs are directed from left to right. Negative supply means demand.
The spannings tree corresponding to the suggested choice of basic variables is illustrated in the right figure below, together with the easily calculated values of these basic variables, i.e the flows in the spanning tree arcs.

```
       Nod 2      Nod 4                          80
    +80 O-----------O -160          +80 O-----------O -160
       /           /                               /
      /           /                               /
     /           /                               / 80
    /           /                               /
   /           /                               /
+120 O-----------O -40           +120 O-----------O -40
    Nod 1      Nod 3                        120
```

The simplex multipliers y_i for the nodes are calculated from $y_4 = 0$ and $y_i - y_j = c_{ij}$ for all ars (i, j) in the spanning tree (left figure below), whereafter the reduced cost for the (only) non-basic variable is calculated from $r_{ij} = c_{ij} - y_i + y_j$ (right figure below).

```
        y2=2        y4=0             y2=2        y4=0
         O-----------O                O           O
              2     /                /
                   /                /
                5 /             6 / r12 = 6-9+2 = -1
                 /                /
          4     /                /
    O-----------O               O           O
   y1=9       y3=5             y1=9        y3=5
```

Since $r_{12} < 0$, we let $x_{12} = t$ and let t increase from 0. Then the values of the basic variables change according to the left figure below. Clearly, t can be increased to at most $t = 80$, and then the new basic solution (spanning tree) in the right figure below is obtained.

```
             80+t                              160
    +80 O-----------O -160          +80 O-----------O -160
       /           /                   /
      /           /                   /
  +t /           / 80-t           80 /
    /           /                   /
   /           /                   /
+120 O-----------O -40          +120 O-----------O -40
        120-t                              40
```

Again, the simplex multipliers y_i for the nodes are calculated from $y_4 = 0$ and $y_i - y_j = c_{ij}$ for all ars (i, j) in the spanning tree (left figure below), whereafter the reduced cost for the (only) non-basic variable is calculated from $r_{ij} = c_{ij} - y_i + y_j$ (right figure below).
Since $r_{34} \geq 0$, the current basic solution $\mathbf{x} = (80, 40, 160, 0)^\mathsf{T}$ is optimal !

```
      y2=2           y4=0                 y2=2           y4=0
       O-----------O                       O              O
      /      2                                           /
     /                                                  /
  6 /                                                5 / r34 = 5-4+0 = 1
   /                                                  /
  /      4                                           /
 O-----------O                           O          O
y1=8        y3=4                        y1=8       y3=4
```

1.(c) We now have a QP problem with equality constraints, i.e. a problem of the form

minimize $\frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} + \mathbf{c}^\mathsf{T}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$,

with $\mathbf{A}$ and $\mathbf{b}$ as above, $\mathbf{c} = \mathbf{0}$, and $\mathbf{H} = \mathbf{I} =$ the 4×4 identity matrix.

We use elementary row operations to transform $\mathbf{A}\mathbf{x} = \mathbf{b}$ to reduced row echelon form:

$$\left[\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & 120 \\ -1 & 0 & 1 & 0 & 80 \\ 0 & -1 & 0 & 1 & -40 \\ 0 & 0 & -1 & -1 & -160 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 80 \\ 0 & 1 & 0 & -1 & 40 \\ 0 & 0 & 1 & 1 & 160 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(Since it is a balanced network flow problem, it is well known that the last row could have been removed already from the start, but that does not change the result.)
From this reduced row echelon form, it follows that the general solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ is obtained by letting $x_{34} = v$ (an arbitrary number), whereafter $x_{12} = 80 - v$, $x_{13} = 40 + v$ and $x_{24} = 160 - v$. Thus, the complete set of solutions to $\mathbf{A}\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x} = \begin{pmatrix} x_{12} \\ x_{13} \\ x_{24} \\ x_{34} \end{pmatrix} = \begin{pmatrix} 80 \\ 40 \\ 160 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} v \;=\; \bar{\mathbf{x}} + \mathbf{z}\, v, \;\; \text{for } v \in I\!R,$$

where $\bar{\mathbf{x}}$ is one solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$, and $\mathbf{z}$ is a basis for the null-space of $\mathbf{A}$.
Changing variables from $\mathbf{x}$ to v leads to a quadratic objective function which is uniquely minimized by the solution $\hat{v}$ to the system $(\mathbf{z}^\mathsf{T}\mathbf{H}\mathbf{z})\, v = -\mathbf{z}^\mathsf{T}(\mathbf{H}\bar{\mathbf{x}} + \mathbf{c})$, provided that $\mathbf{z}^\mathsf{T}\mathbf{H}\mathbf{z}$ is positive definite (> 0 in this one-variable case).

We get that $\mathbf{z}^\mathsf{T}\mathbf{H}\mathbf{z} = \mathbf{z}^\mathsf{T}\mathbf{z} = 4 > 0$ and $-\mathbf{z}^\mathsf{T}(\mathbf{H}\bar{\mathbf{x}} + \mathbf{c}) = -\mathbf{z}^\mathsf{T}\bar{\mathbf{x}} = 200$,
so the unique solution to the system above is $\hat{v} = 200/4 = 50$,
and the unique global optimal solution to the considered QP problem is

$$\hat{\mathbf{x}} = \bar{\mathbf{x}} + \mathbf{z}\,\hat{v} = \begin{pmatrix} 80 \\ 40 \\ 160 \\ 0 \end{pmatrix} + \begin{pmatrix} -50 \\ 50 \\ -50 \\ 50 \end{pmatrix} = \begin{pmatrix} 30 \\ 90 \\ 110 \\ 50 \end{pmatrix}.$$

2.(a) We have an LP problem on the standard form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}\,, \\ & \mathbf{x} \geq \mathbf{0}\,, \end{aligned}$$

where $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ and $\mathbf{c}^\mathsf{T} = (3,\, c,\, c,\, -1)$.

There are six different ways to chose two of the four columns in $\mathbf{A}$:

$\beta = (1,2)$ with $\mathbf{A}_\beta = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\beta = (1,3)$ with $\mathbf{A}_\beta = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$,

$\beta = (1,4)$ with $\mathbf{A}_\beta = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $\beta = (2,3)$ with $\mathbf{A}_\beta = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$,

$\beta = (2,4)$ with $\mathbf{A}_\beta = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$, $\beta = (3,4)$ with $\mathbf{A}_\beta = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$.

But the matrices $\mathbf{A}_\beta$ corresponding to $\beta = (1,4)$ and $\beta = (2,3)$ are singular, so they are not basic matrices and do not correspond to basic solutions.

For each of the remaining four choices, the values of the basic variables are $\mathbf{x}_\beta = \bar{\mathbf{b}}$, where the vector $\bar{\mathbf{b}}$ is calculated from the system $\mathbf{A}_\beta \bar{\mathbf{b}} = \mathbf{b}$.
Straight forward calculations give the following:

$$\mathbf{A}_\beta = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \bar{\mathbf{b}} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \quad \mathbf{A}_\beta = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow \bar{\mathbf{b}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix},$$

$$\mathbf{A}_\beta = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \Rightarrow \bar{\mathbf{b}} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}, \quad \mathbf{A}_\beta = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$$

Thus, there is only one basic feasible solution, namely $\mathbf{x} = (5,0,2,0)^\mathsf{T}$, which corresponds to $\beta = (1,3)$ and $\nu = (2,4)$.

2.(b) Now $\mathbf{c}^\mathsf{T} = (3,1,1,-1)$.

The vector $\mathbf{x} = (5,0,2,0)^\mathsf{T}$ is the basic solution corresponding to $\beta = (1,3)$ and $\nu = (2,4)$ with $\mathbf{A}_\beta = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{A}_\nu = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ and $\bar{\mathbf{b}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

The vector $\mathbf{y}$ with simplex multipliers is obtained from the system $\mathbf{A}_\beta^\mathsf{T}\mathbf{y} = \mathbf{c}_\beta$, i.e. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, with the solution $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Then the reduced costs for the non-basic variables are obtained from

$$\mathbf{r}_\nu^\mathsf{T} = \mathbf{c}_\nu^\mathsf{T} - \mathbf{y}^\mathsf{T}\mathbf{A}_\nu = (1,-1) - (1,\, 2) \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = (2,\, 2).$$

For every feasible solution, the objective value may now be expressed as $z = \bar{z} + r_{\nu_1}x_{\nu_1} + r_{\nu_2}x_{\nu_2} = \bar{z} + r_2x_2 + r_4x_4 = 17 + 2x_2 + 2x_4$, which is > 17 for all feasible solutions except for $\mathbf{x} = (5,0,2,0)^\mathsf{T}$, which has $z = \bar{z} = 17$.
Thus, $\mathbf{x} = (5,0,2,0)^\mathsf{T}$ is the unique optimal solution when $c = 1$.

2.(c) Now $\mathbf{c}^\mathsf{T} = (3, -1, -1, -1)$.

We start the simplex method from the above BFS, i.e. $\mathbf{x} = (5, 0, 2, 0)^\mathsf{T}$, corresponding to $\beta = (1, 3)$ and $\nu = (2, 4)$ with $\mathbf{A}_\beta = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{A}_\nu = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ and $\bar{\mathbf{b}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

The vector $\mathbf{y}$ with simplex multipliers is obtained from the system $\mathbf{A}_\beta^\mathsf{T}\mathbf{y} = \mathbf{c}_\beta$,
which now become $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, with the solution $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Then the reduced costs for the non-basic variables are obtained from

$\mathbf{r}_\nu^\mathsf{T} = \mathbf{c}_\nu^\mathsf{T} - \mathbf{y}^\mathsf{T}\mathbf{A}_\nu = (-1, -1) - (2, 1)\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = (-2,\ 2).$

Since $r_{\nu_1} = r_2 = -2$ is smallest, and < 0, we let x_2 increase from zero.

Then we should calculate the vector $\bar{\mathbf{a}}_2$ from the system $\mathbf{A}_\beta\bar{\mathbf{a}}_2 = \mathbf{a}_2$, i.e.

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} \bar{a}_{12} \\ \bar{a}_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, with the solution $\bar{\mathbf{a}}_2 = \begin{pmatrix} \bar{a}_{12} \\ \bar{a}_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Since $\bar{\mathbf{a}}_2 \le \mathbf{0}$, the simplex method stops here, with the conclusion that there is no optimal solution to the problem.

If $x_2 = t > 0$ and $x_4 = 0$, the objective value becomes $z = \bar{z} + r_2 t = 13 - 2t$, while $\mathbf{x}_\beta = \bar{\mathbf{b}} - \bar{\mathbf{a}}_2 t$, i.e. $x_1 = 5$ and $x_3 = 2 + t$.

By chosing e.g. $t = 1000$ we obtain the feasible solution $\mathbf{x} = (5, 2000, 2002, 0)^\mathsf{T}$ with objective value $z = 13 - 2000 = -1987 < -1000$.

2.(d) Now $\mathbf{c}^\mathsf{T} = (3, 0, 0, -1)$.

As in 2.(c), we start the simplex method from the BFS in 2.(b).

The vector $\mathbf{y}$ with simplex multipliers is obtained from the system $\mathbf{A}_\beta^\mathsf{T}\mathbf{y} = \mathbf{c}_\beta$,
which now become $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, with the solution $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$.

Then the reduced costs for the non-basic variables are obtained from

$\mathbf{r}_\nu^\mathsf{T} = \mathbf{c}_\nu^\mathsf{T} - \mathbf{y}^\mathsf{T}\mathbf{A}_\nu = (0, -1) - (1.5,\ 1.5)\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = (0,\ 2).$

Since $\mathbf{r}_\nu \ge \mathbf{0}$, the current basic feasible solution $\mathbf{x} = (5, 0, 2, 0)^\mathsf{T}$ is an optimal solution, now with optimal value $\mathbf{c}^\mathsf{T}\mathbf{x} = 15$.

But since $r_{\nu_1} = r_2 = 0$ there may also be other optimal solutions, obtained by increasing x_2 from zero. The vector $\bar{\mathbf{a}}_2$ is the same as in 2.(c), so if $x_2 = t > 0$ and $x_4 = 0$, the objective value becomes $z = \bar{z} + r_2 t = 15 + 0\,t = 15$ (independent of t), while $\mathbf{x}_\beta = \bar{\mathbf{b}} - \bar{\mathbf{a}}_2 t$, i.e. $x_1 = 5$ and $x_3 = 2 + t$ (as in 2.(c)).

The conclusion is that $\mathbf{x}(t) = (5,\ t,\ 2{+}t,\ 0)^\mathsf{T}$ is an optimal solution for any $t \ge 0$, with the optimal value $\mathbf{c}^\mathsf{T}\mathbf{x}(t) = 15$. Three examples of optimal solutions are $\mathbf{x}(0) = (5,\ 0,\ 2,\ 0)^\mathsf{T}$, $\mathbf{x}(1) = (5,\ 1,\ 3,\ 0)^\mathsf{T}$ and $\mathbf{x}(2) = (5,\ 2,\ 4,\ 0)^\mathsf{T}$.

2.(e) If the primal problem is on the standard form

$$\begin{array}{rl} \text{minimize} & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

the corresponding dual problem is: maximize $\mathbf{b}^\mathsf{T}\mathbf{y}$ subject to $\mathbf{A}^\mathsf{T}\mathbf{y} \leq \mathbf{c}$, which becomes

$$\begin{array}{rrcrcr} \text{maximize} & 3y_1 & + & 7y_2 & & \\ \text{subject to} & y_1 & + & y_2 & \leq & 3, \\ & y_1 & - & y_2 & \leq & c, \\ & -y_1 & + & y_2 & \leq & c, \\ & -y_1 & - & y_2 & \leq & -1. \end{array}$$

The dual problems for various c can be illustrated by drawing the constraints and some level lines for the dual objective function in a coordinate system with the dual variables y_1 and y_2 on the axes. (Figures omitted here.)

If $c = 1$, the feasible region becomes a rectangle (in fact a square) with corners $(1, 0)^\mathsf{T}$, $(0, 1)^\mathsf{T}$, $(1, 2)^\mathsf{T}$ and $(2, 1)^\mathsf{T}$.
From the level lines it is easily seen that the corner $\mathbf{y} = (1, 2)^\mathsf{T}$ is the optimal solution, with objective value $3y_1 + 7y_2 = 17$.
(This $\mathbf{y}$ is also the vector $\mathbf{y}$ of "simplex multipliers" from 2.(b)).

If $c = -1$, the feasible region becomes empty, since there is no point which satisfies both $y_1 - y_2 \leq -1$ and $-y_1 + y_2 \leq -1$. This is consistent with the fact that there was no optimal solution to the primal problem in 2.(c).

If $c = 0$, the feasible region becomes the line segment between the points $(0.5, 0.5)^\mathsf{T}$ and $(1.5, 1.5)^\mathsf{T}$.
From the level lines it is easily seen that the end point $\mathbf{y} = (1.5, 1.5)^\mathsf{T}$ is the optimal solution, with objective value $3y_1 + 7y_2 = 15$.
(This $\mathbf{y}$ is also the vector $\mathbf{y}$ of "simplex multipliers" from 2.(d)).

3.(a) The objective function is $f(\mathbf{x}) = (x_1 - x_2)^3 + (x_1 - x_2)^2 + (x_2 - 1)^2$.
Then the gradient of f becomes: $\nabla f(\mathbf{x})^\mathsf{T} = \begin{pmatrix} 3(x_1-x_2)^2 + 2(x_1-x_2) \\ -3(x_1-x_2)^2 - 2(x_1-x_2) + 2(x_2-1) \end{pmatrix}$,
while the Hessian of f becomes: $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 6(x_1-x_2)+2 & -6(x_1-x_2)-2 \\ -6(x_1-x_2)-2 & 6(x_1-x_2)+4 \end{bmatrix}$.
The starting point is $\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, with $f(\mathbf{x}^{(1)}) = 1$.
$\mathbf{F}(\mathbf{x}^{(1)}) = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$ is positive definite since $2 > 0$, $4 > 0$ and $2 \cdot 4 - (-2)\cdot(-2) > 0$.

Then the first Newton search direction $\mathbf{d}^{(1)}$ is obtained by solving the system
$\mathbf{F}(\mathbf{x}^{(1)})\mathbf{d} = -\nabla f(\mathbf{x}^{(1)})^\mathsf{T}$, i.e. $\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \mathbf{d} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, with the solution $\mathbf{d}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
First try $t_1 = 1$, so that $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + t_1 \mathbf{d}^{(1)} = \mathbf{x}^{(1)} + \mathbf{d}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Then $f(\mathbf{x}^{(2)}) = 0 < f(\mathbf{x}^{(1)})$, so $t_1 = 1$ is accepted, and the first iteration is completed.

3.(b) Any local optimal solution to the problem of minimizing $f(\mathbf{x})$ without constraints
must satisfy that $\nabla f(\mathbf{x})^\mathsf{T} = \begin{pmatrix} 3(x_1-x_2)^2 + 2(x_1-x_2) \\ -3(x_1-x_2)^2 - 2(x_1-x_2) + 2(x_2-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
By adding these two equations, we obtain that $x_2 = 1$.
By plugging this into any of the two equations, we obtain that $3(x_1-1)^2 + 2(x_1-1) =$
$= (x_1-1)(3(x_1-1)+2) = 0$, with the solutions $x_1 = 1$ or $x_1 = 1/3$.
Thus, the only solutions to $\nabla f(\mathbf{x})^\mathsf{T} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are $\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\bar{\mathbf{x}} = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$.
$\mathbf{F}(\hat{\mathbf{x}}) = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$ is positive definite, so $\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a local optimal solution, while

$\mathbf{F}(\bar{\mathbf{x}}) = \begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix}$ is not positive semidefinite, so $\bar{\mathbf{x}} = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$ is *not* a local optimal solution.

3.(c) There are several ways to see that there is no global optimal solution to the problem.
One is to let x_1 be fixed to zero. Then $f(\mathbf{x}) = -x_2^3 + x_2^2 + (x_2 - 1)^2 \to -\infty$ when $x_2 \to \infty$.

3.(d) Let C be a given convex set in $\mathbb{R}^2$.
If the Hessian $\mathbf{F}(\mathbf{x})$ is positive semidefinite for all $\mathbf{x} \in C$ then f is convex on C.
$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 6(x_1-x_2)+2 & -6(x_1-x_2)-2 \\ -6(x_1-x_2)-2 & 6(x_1-x_2)+4 \end{bmatrix}$ is positive semidefinite if and only if:

(i) $6(x_1-x_2) + 2 \geq 0$,
(ii) $6(x_1-x_2) + 4 \geq 0$, and
(iii) $(6(x_1-x_2)+2)(6(x_1-x_2)+4) - (6(x_1-x_2)+2)^2 \geq 0$.

(i) holds if $6(x_1-x_2) + 2 \geq 0$, but then (ii) and (iii) also hold!
It follows that with $a_1 = -3$ and $a_2 = 3$, f is convex on the convex set
$C = \{\, (x_1, x_2)^\mathsf{T} \in \mathbb{R}^2 \mid a_1 x_1 + a_2 x_2 \leq 1 \}$.

4.(a) The Lagrange function for the considered problem is given by

$L(\mathbf{x},\mathbf{y}) = \frac{1}{2}\,\mathbf{x}^\mathsf{T}\mathbf{x} + \mathbf{y}^\mathsf{T}(\mathbf{b} - \mathbf{A}\mathbf{x})$, with $\mathbf{x} \in I\!R^4$ and $y \in I\!R^2$.

The Lagrange relaxed problem $\mathrm{PR}_\mathbf{y}$ is defined, for a given $\mathbf{y} \geq \mathbf{0}$, as the problem of minimizing $L(\mathbf{x},\mathbf{y})$ with respect to $\mathbf{x} \in I\!R^4$.

Since $L(\mathbf{x},\mathbf{y}) = \frac{1}{2}\,\mathbf{x}^\mathsf{T}\mathbf{I}\,\mathbf{x} - (\mathbf{A}^\mathsf{T}\mathbf{y})^\mathsf{T}\mathbf{x} + \mathbf{b}^\mathsf{T}\mathbf{y}$, and the unit matrix $\mathbf{I}$ is positive definite, the unique optimal solution to $\mathrm{PR}_\mathbf{y}$ is given by $\tilde{\mathbf{x}}(\mathbf{y}) = \mathbf{A}^\mathsf{T}\mathbf{y}$.

Then the dual objective function becomes $\varphi(\mathbf{y}) = L(\tilde{\mathbf{x}}(\mathbf{y}),\mathbf{y}) = -\frac{1}{2}\,\mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{A}^\mathsf{T}\mathbf{y} + \mathbf{b}^\mathsf{T}\mathbf{y}$,

where $\mathbf{A}\mathbf{A}^\mathsf{T} = \begin{bmatrix} 30 & 20 \\ 20 & 30 \end{bmatrix}$, which is positive definite, and $\mathbf{b} = \begin{pmatrix} 18 \\ 30 \end{pmatrix}$.

The dual problem can thus be written:
D: maximize $\varphi(\mathbf{y}) = -15y_1^2 - 20y_1y_2 - 15y_2^2 + 18y_1 + 30y_2$ subject to $y_1 \geq 0$ and $y_2 \geq 0$.

4.(b) In order for $(\hat{\mathbf{x}},\hat{\mathbf{y}})$ to satisfy the first of the conditions in GOC, $\hat{\mathbf{x}}$ should minimize $L(\mathbf{x},\hat{\mathbf{y}})$ with respect to $\mathbf{x} \in I\!R^4$, which according to (a) gives that $\hat{\mathbf{x}} = \tilde{\mathbf{x}}(\hat{\mathbf{y}}) = \mathbf{A}^\mathsf{T}\hat{\mathbf{y}}$, i.e.

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} = \begin{pmatrix} \hat{y}_1 + 4\hat{y}_2 \\ 2\hat{y}_1 + 3\hat{y}_2 \\ 3\hat{y}_1 + 2\hat{y}_2 \\ 4\hat{y}_1 + \hat{y}_2 \end{pmatrix}.$$

Then the global optimality conditions (GOC) become:

(GOC-1): $(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4)^\mathsf{T} = \hat{y}_1(1,2,3,4)^\mathsf{T} + \hat{y}_2(4,3,2,1)^\mathsf{T}$.
(GOC-2): $18-\hat{x}_1-2\hat{x}_2-3\hat{x}_3-4\hat{x}_4 \leq 0$ and $30-4\hat{x}_1-3\hat{x}_2-2\hat{x}_3-\hat{x}_4 \leq 0$.
(GOC-3): $\hat{y}_1 \geq 0$ and $\hat{y}_2 \geq 0$.
(GOC-4): $\hat{y}_1(18-\hat{x}_1-2\hat{x}_2-3\hat{x}_3-4\hat{x}_4) = 0$ and $\hat{y}_2(30-4\hat{x}_1-3\hat{x}_2-2\hat{x}_3-\hat{x}_4) = 0$.

We are searching for a solution with $\hat{y}_1 = 0$ and $\hat{y}_2 > 0$.
In this case, the above conditions are simplified to the following:

(GOC-1)': $(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4)^\mathsf{T} = \hat{y}_2(4,3,2,1)^\mathsf{T}$.
(GOC-2)': $18-\hat{x}_1-2\hat{x}_2-3\hat{x}_3-4\hat{x}_4 \leq 0$ and $30-4\hat{x}_1-3\hat{x}_2-2\hat{x}_3-\hat{x}_4 \leq 0$.
(GOC-3)': $\hat{y}_1 = 0$ and $\hat{y}_2 > 0$.
(GOC-4)': $30-4\hat{x}_1-3\hat{x}_2-2\hat{x}_3-\hat{x}_4 = 0$.

Combining (GOC-1)' and (GOC-4)' implies that $\hat{y}_2 = 1$ and $(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4)^\mathsf{T} = (4,3,2,1)^\mathsf{T}$, which satisfy both (GOC-1)' and (GOC-4)'.

Then checking (GOC-2)' and (GOC-3)' gives that:
$18-\hat{x}_1-2\hat{x}_2-3\hat{x}_3-4\hat{x}_4 = -2 \leq 0$, OK!, and $\hat{y}_2 = 1 > 0$, OK!

It follows that the GOC are satisfied by $\hat{\mathbf{x}} = (4,3,2,1)^\mathsf{T}$ and $\hat{\mathbf{y}} = (0,1)^\mathsf{T}$, and consequently, $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are optimal to P and D, respectively.

5.(a) That $f(\hat{\mathbf{x}}) \le f(\mathbf{x})$ for all $\mathbf{x} \in I\!R^n$ means by definition that $\hat{\mathbf{x}}$ is a global minimum point to the quadratic function f, which in turn (since $\mathbf{H}$ is symmetric and positive semidefinite) is equivalent to that $\mathbf{H}\hat{\mathbf{x}} = -\mathbf{c}$.

This implies that $\hat{\mathbf{x}} \neq \mathbf{0}$, since if $\hat{\mathbf{x}} = \mathbf{0}$ then $\mathbf{H}\hat{\mathbf{x}} = \mathbf{H0} = \mathbf{0} \neq -\mathbf{c}$ (since $\mathbf{c} \neq \mathbf{0}$).

Further, since $\hat{\mathbf{x}}$ is a global minimum point to f, while $\mathbf{0}$ is not, we get that

$f(\hat{\mathbf{x}}) < f(\mathbf{0}) = \frac{1}{2}\,\mathbf{0}^\mathsf{T}\mathbf{H}\,\mathbf{0} + \mathbf{c}^\mathsf{T}\mathbf{0} = 0.$

Finally, since

$f(\hat{\mathbf{x}}) = \frac{1}{2}\,\hat{\mathbf{x}}^\mathsf{T}\mathbf{H}\hat{\mathbf{x}} + \mathbf{c}^\mathsf{T}\hat{\mathbf{x}} = \frac{1}{2}\,\hat{\mathbf{x}}^\mathsf{T}\mathbf{H}\hat{\mathbf{x}} + (-\mathbf{H}\hat{\mathbf{x}})^\mathsf{T}\hat{\mathbf{x}} = \frac{1}{2}\,\hat{\mathbf{x}}^\mathsf{T}\mathbf{H}\hat{\mathbf{x}} - \hat{\mathbf{x}}^\mathsf{T}\mathbf{H}\hat{\mathbf{x}} = -\frac{1}{2}\,\hat{\mathbf{x}}^\mathsf{T}\mathbf{H}\hat{\mathbf{x}},$

we get that $0 > f(\hat{\mathbf{x}}) = -\frac{1}{2}\,\hat{\mathbf{x}}^\mathsf{T}\mathbf{H}\hat{\mathbf{x}}$, which shows that $\hat{\mathbf{x}}^\mathsf{T}\mathbf{H}\hat{\mathbf{x}} > 0$.

5.(b) The considered problem can be written:

minimize $\frac{1}{2}\,\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} + \mathbf{c}^\mathsf{T}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$, with $\mathbf{A} = \mathbf{c}^\mathsf{T}$ and $\mathbf{b} = k\,\mathbf{c}^\mathsf{T}\hat{\mathbf{x}}$ (a scalar).

The matrix $\mathbf{H}$ is positive semidefinite, so the following Lagrange optimality conditions are both necessary and sufficient conditions for a global optimum:

$$\begin{aligned} \mathbf{H}\mathbf{x} - \mathbf{A}^\mathsf{T}\mathbf{u} &= -\mathbf{c} \\ \mathbf{A}\mathbf{x} \qquad &= \quad \mathbf{b} \end{aligned}$$

In our case, $\mathbf{H}\mathbf{x} - \mathbf{A}^\mathsf{T}\mathbf{u} = -\mathbf{c}$ is equivalent to that $\mathbf{H}\mathbf{x} = -(1-u)\mathbf{c}$, with $u \in I\!R$, while $\mathbf{A}\mathbf{x} = \mathbf{b}$ is equivalent to that $\mathbf{c}^\mathsf{T}\mathbf{x} = k\,\mathbf{c}^\mathsf{T}\hat{\mathbf{x}}$.

From 5.(a), we know that $\mathbf{H}\hat{\mathbf{x}} = -\mathbf{c}$, so the vector $\bar{\mathbf{x}} = (1-u)\hat{\mathbf{x}}$ satisfies $\mathbf{H}\bar{\mathbf{x}} = -(1-u)\mathbf{c}$. Further, by chosing $u = 1-k$, so that $(1-u) = k$, $\bar{\mathbf{x}}$ also satisfies $\mathbf{c}^\mathsf{T}\bar{\mathbf{x}} = k\,\mathbf{c}^\mathsf{T}\hat{\mathbf{x}}$.

The conclusion is that $\bar{\mathbf{x}} = k\,\hat{\mathbf{x}}$ is a global optimal solution to the considered problem.

The optimal value of the considered problem is now given by

$f(\bar{\mathbf{x}}) = \frac{1}{2}\,\bar{\mathbf{x}}^\mathsf{T}\mathbf{H}\bar{\mathbf{x}} + \mathbf{c}^\mathsf{T}\bar{\mathbf{x}} = \frac{1}{2}\,k^2\hat{\mathbf{x}}^\mathsf{T}\mathbf{H}\hat{\mathbf{x}} + k(-\mathbf{H}\hat{\mathbf{x}})^\mathsf{T}\hat{\mathbf{x}} = \frac{1}{2}(k^2-2k)\,\hat{\mathbf{x}}^\mathsf{T}\mathbf{H}\hat{\mathbf{x}}.$

Since, by 5.(a), $\hat{\mathbf{x}}^\mathsf{T}\mathbf{H}\hat{\mathbf{x}} > 0$, we get that

$f(\bar{\mathbf{x}}) > 0$ if and only if $k^2-2k > 0$, i.e. if and only if $k > 2$ or $k < 0$.