Solutions to exam in SF1811 Optimization, June 3, 2014

1.(a)

The considered problem may be modelled as a minimum-cost network flow problem with six nodes F1, F2, K1, K2, K3, K4, here called 1,2,3,4,5,6,
and eight arcs (F1,K1), (F1,K2), (F1,K3), (F1,K4), (F2,K1), (F2,K2), (F2,K3), (F2,K4), here called (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6).

The suggested solution corresponds to a spanning tree with the arcs (1,3), (1,4), (2,4), (2,5) och (2,6), i.e. a basic solution. It is a feasible basic solution since all the balance equations (in all nodes) are satisfied and all variables are non-negative.

The simplex variables y_i are obtained from the equations $y_i - y_j = c_{ij}$ for basic variables, and $y_6 = 0$. This gives $y = (5,\, 3, -2, -1, -2,\, 0)$.
Then the reduced costs for the non-basic variables are obtained from $r_{ij} = c_{ij} - y_i + y_j$.

This gives: $r_{15} = 8 - 5 + (-2) = 1,\;\; r_{16} = 6 - 5 + 0 = 1,\;\; r_{23} = 6 - 3 + (-2) = 1.$

Since all $r_{ij} \geq 0$, the suggested solution is optimal.

1.(b)

The vector $\hat{\mathbf{x}}$ is a global optimal solution to the problem of minimizing the function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x}$ if and only if $\mathbf{H}$ is positive semi-definite and $\mathbf{H}\hat{\mathbf{x}} = \mathbf{0}$.

To check if $\mathbf{H}$ is positive semi-definite, we try to LDL$^\mathsf{T}$-factorize $\mathbf{H}$.

$$\mathbf{H} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & k \end{bmatrix}.$$

Add +1 times row 1 to row 2. Then add +1 times column 1 to column 2. This gives

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{E}_1\mathbf{H}\mathbf{E}_1^\mathsf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & k \end{bmatrix}.$$

Add +1 times row 2 to row 3. Then add +1 times column 2 to column 3. This gives

$$\mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{E}_2\mathbf{E}_1\mathbf{H}\mathbf{E}_1^\mathsf{T}\mathbf{E}_2^\mathsf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k-1 \end{bmatrix}.$$

The LDL$^\mathsf{T}$-factorization is ready, and one gets

$$\mathbf{H} = \mathbf{L}\mathbf{D}\mathbf{L}^\mathsf{T} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k-1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

From this, it follows that $\mathbf{H}$ is positive semi-definite if and only if $k \geq 1$.
The system $\mathbf{H}\mathbf{x} = \mathbf{0}$ always has at least the solution $\mathbf{x} = \mathbf{0}$ (the trivial solution), and it has an infinite number of solutions if and only if $\mathbf{H}$ is singular.
From above, $\mathbf{H}$ is both positive semi-definite and singular if and only if $k = 1$. Then

$$\mathbf{H} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Some calculations give that the set of solutions to $\mathbf{H}\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = (t,\, t,\, t)^\mathsf{T}$ for $t \in \mathbb{R}$, and this is then the set of global optimal solutions to the problem of minimizing $\frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x}$.

2.(a) (Sorry for the Swedish text)

Inför slackvariabler x_4, x_5 och x_6 så att problemet blir på standardformen

$$\begin{aligned} \text{minimera} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{då} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

där $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ och $\mathbf{c} = (1,\ 1,\ 4,\ 0,\ 0,\ 0)^\mathsf{T}$.

I startlösningen ska enligt uppgiftslydelsen x_1, x_2 och x_3 vara basvariabler, dvs $\beta = (1, 2, 3)$ och $\delta = (4, 5, 6)$.

Motsvarande basmatris ges av $\mathbf{A}_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Basvariablernas värden i baslösningen ges av $\mathbf{x}_\beta = \bar{\mathbf{b}}$, där vektorn $\bar{\mathbf{b}}$ beräknas ur ekvationssystemet $\mathbf{A}_\beta \bar{\mathbf{b}} = \mathbf{b}$,

dvs $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, med lösningen $\bar{\mathbf{b}} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Här använde vi den givna räknehjälpen.

Vektorn $\mathbf{y}$ med simplexmultiplikatorerna värden erhålls ur systemet $\mathbf{A}_\beta^\mathsf{T} \mathbf{y} = \mathbf{c}_\beta$,

dvs $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$, med lösningen $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

Här använde vi igen den givna räknehjälpen.

Reducerade kostnaderna för icke-basvariablerna ges av $\mathbf{r}_\delta^\mathsf{T} = \mathbf{c}_\delta - \mathbf{y}^\mathsf{T} \mathbf{A}_\delta =$

$= (0,\ 0,\ 0) - (-1,\ 2,\ 2) \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-1,\ 2,\ 2)$.

Eftersom $r_{\delta_1} = r_4 = -1$ är minst, och < 0, ska vi låta x_4 bli ny basvariabel.

Då behöver vi beräkna vektorn $\bar{\mathbf{a}}_4$ ur systemet $\mathbf{A}_\beta \bar{\mathbf{a}}_4 = \mathbf{a}_4$,

dvs $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} \bar{a}_{14} \\ \bar{a}_{24} \\ \bar{a}_{34} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, med lösningen $\bar{\mathbf{a}}_4 = \begin{pmatrix} \bar{a}_{14} \\ \bar{a}_{24} \\ \bar{a}_{34} \end{pmatrix} = \begin{pmatrix} -0.5 \\ -0.5 \\ 0.5 \end{pmatrix}$.

Här använde vi igen den givna räknehjälpen.

Det största värde som den nya basvariabeln x_4 kan ökas till ges av

$$t^{\max} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{i4}} \mid \bar{a}_{i4} > 0 \right\} = \frac{\bar{b}_3}{\bar{a}_{34}} = \frac{2}{0.5}\ .$$

Minimerande index är $i = 3$, varför $x_{\beta_3} = x_3$ inte längre får vara kvar som basvariabel.

Nu är alltså $\beta = (1, 2, 4)$ och $\delta = (3, 5, 6)$.

Motsvarande basmatris ges av $\mathbf{A}_\beta = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Basvariablernas värden i baslösningen ges av $\mathbf{x}_\beta = \bar{\mathbf{b}}$, där vektorn $\bar{\mathbf{b}}$ beräknas ur ekvationssystemet $\mathbf{A}_\beta \bar{\mathbf{b}} = \mathbf{b}$,

dvs $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, med lösningen $\bar{\mathbf{b}} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

Vektorn $\mathbf{y}$ med simplexmultiplikatorernas värden erhålls ur systemet $\mathbf{A}_\beta^\mathsf{T} \mathbf{y} = \mathbf{c}_\beta$,

dvs $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, med lösningen $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Reducerade kostnaderna för icke-basvariablerna ges av $\mathbf{r}_\delta^\mathsf{T} = \mathbf{c}_\delta - \mathbf{y}^\mathsf{T} \mathbf{A}_\delta =$

$= (4,\ 0,\ 0) - (0,\ 1,\ 1) \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = (2,\ 1,\ 1)$.

Eftersom $\mathbf{r}_\delta \geq \mathbf{0}$ så är den aktuella baslösningen optimal.
Alltså är punkten $x_1 = 2$, $x_2 = 2$, $x_3 = 0$, $x_4 = 2$, $x_5 = 0$, $x_6 = 0$ optimal.
Optimalvärdet är $\mathbf{c}^\mathsf{T} \mathbf{x} = 4$.

2.(b)
If the primal problem is on the standard form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\mathsf{T} \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

then the dual problem is on the form

$$\text{maximize } \mathbf{b}^\mathsf{T} \mathbf{y} \text{ subject to } \mathbf{A}^\mathsf{T} \mathbf{y} \leq \mathbf{c},$$

which for the current example becomes

$$\begin{aligned} \text{maximize} \quad & 2y_1 + 2y_2 + 2y_3 \\ \text{subject to} \quad & y_1 + y_2 \leq 1, \\ & y_1 + y_3 \leq 1, \\ & y_2 + y_3 \leq 4, \\ & -y_1 \leq 0, \\ & -y_2 \leq 0, \\ & -y_3 \leq 0. \end{aligned}$$

It is well known that an optimal solution to this dual problem is given by the vector $\mathbf{y}$ of simplex multipliers in the optimal basic solution from 2.(a), i.e. $\mathbf{y} = (0,\ 1,\ 1)^\mathsf{T}$.
We note that this is a feasible solution to the dual problem with $\mathbf{b}^\mathsf{T} \mathbf{y} = 4 =$ the optimal value of the primal problem.

3.

The gradient of f is $\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$, where $\frac{\partial f}{\partial x_j} = 4x_j^3 + 3x_j^2 + 2x_j + 1$.

The Hessian of f is $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & 0 & 0 \\ 0 & \frac{\partial^2 f}{\partial x_2^2} & 0 \\ 0 & 0 & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$, where $\frac{\partial^2 f}{\partial x_j^2} = 12x_j^2 + 6x_j + 2$.

(b) f is convex on $\mathbb{R}^3$ if and only if $\mathbf{F}(\mathbf{x})$ is positive semi-definite for all $\mathbf{x} \in \mathbb{R}^3$.
A diagonal matrix is positive semi-definite if and only if all diagonal elements are ≥ 0.

But $\frac{\partial^2 f}{\partial x_j^2} = 12(x_j^2 + \frac{1}{2}x_j + \frac{1}{6}) = 12((x_j + \frac{1}{4})^2 - \frac{1}{16} + \frac{1}{6}) > 0$,
so $\mathbf{F}(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \mathbb{R}^3$. Thus, f is (strictly) convex on $\mathbb{R}^3$.

(a) The given starting point is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

If $\mathbf{F}(\mathbf{x}^{(1)})$ is positive definite, which it is according to above, then the first Newton direction $\mathbf{d}^{(1)}$ is obtained as the solution to the system $\mathbf{F}(\mathbf{x}^{(1)})\mathbf{d} = -\nabla f(\mathbf{x}^{(1)})^\mathsf{T}$, i.e.

$\begin{bmatrix} 20 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \mathbf{d} = -\begin{pmatrix} 10 \\ 1 \\ -2 \end{pmatrix}$, with the unique solution $\mathbf{d}^{(1)} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1/4 \end{pmatrix}$.

We try first with $t_1 = 1$, so that $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + t_1\mathbf{d}^{(1)} = \mathbf{x}^{(1)} + \mathbf{d}^{(1)} = \begin{pmatrix} 1/2 \\ -1/2 \\ -3/4 \end{pmatrix}$.

Then $f(\mathbf{x}^{(2)}) = 85/256 < 4 = f(\mathbf{x}^{(1)})$, so $t_1 = 1$ is accepted.

Now we have made a complete iteration with Newtons method and obtained the next iteration point $\mathbf{x}^{(2)} = \begin{pmatrix} 1/2 \\ -1/2 \\ -3/4 \end{pmatrix}$ with $f(\mathbf{x}^{(2)}) = 85/256$.

(c) Here comes a possible lower bound on $f(\mathbf{x})$.

First, we note that $x_j^2 + x_j = (x_j + \frac{1}{2})^2 - \frac{1}{4} \geq -\frac{1}{4}$.

From this, it follows that $x_j^4 + x_j^3 = x_j^2(x_j^2 + x_j) \geq -\frac{1}{4}\,x_j^2$, from which we get that

$x_j^4 + x_j^3 + x_j^2 + x_j \geq -\frac{1}{4}\,x_j^2 + x_j^2 + x_j = \frac{3}{4}(x_j^2 + \frac{4}{3}x_j) = \frac{3}{4}((x_j + \frac{2}{3})^2 - \frac{4}{9}) \geq= \frac{3}{4}(-\frac{4}{9}) = -\frac{1}{3}$.

This holds for each j, so we obtain that

$$f(\mathbf{x}) = \sum_{j=1}^{3}(x_j^4 + x_j^3 + x_j^2 + x_j) \geq -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} = -1.$$

Thus, $L = -1$ is a lower bound on $f(\mathbf{x})$, but possibly not the highest one.

4.(a)

The perimeter problem can be written as the following minimization problem:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = -4x_1 - 4x_2 \\ \text{subject to} \quad & h(\mathbf{x}) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} - 1 = 0. \end{aligned}$$

We assume that x_1 and x_2 are > 0, but don't formulate that explicitly in the optimality conditions, it will naturally be satisfied anyhow by the optimal solution.
The Lagrange conditions for the problem become:

$$\begin{aligned} -4 + \frac{2\lambda x_1}{a_1^2} &= 0, \\ -4 + \frac{2\lambda x_2}{a_2^2} &= 0, \\ \frac{x_1^2}{a^2} + \frac{x_2^2}{a_2^2} - 1 &= 0. \end{aligned}$$

Since every feasible point is also a regular point (the gradient of the only constraint function is never the zero vector) the Lagrange conditions are necessary conditons for an optimal solution. Some calculation give that the only solution (with $x_1 > 0$ and $x_2 > 0$) is
$x_1 = \dfrac{a_1^2}{\sqrt{a_1^2 + a_2^2}}$, $x_2 = \dfrac{a_2^2}{\sqrt{a_1^2 + a_2^2}}$, $u = 2\sqrt{a_1^2 + a_2^2}$, with perimeter $4x_1 + 4x_2 = 4\sqrt{a_1^2 + a_2^2}$.

4.(b)

The area problem can be written as the following minimization problem:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = -4x_1 x_2 \\ \text{subject to} \quad & h(\mathbf{x}) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} - 1 = 0. \end{aligned}$$

Again, we assume that x_1 and x_2 are > 0, but don't formulate that explicitly in the optimality conditions. The Lagrange conditions for the problem become:

$$\begin{aligned} -4x_2 + \frac{2\lambda x_1}{a_1^2} &= 0, \\ -4x_1 + \frac{2\lambda x_2}{a_2^2} &= 0, \\ \frac{x_1^2}{a^2} + \frac{x_2^2}{a_2^2} - 1 &= 0. \end{aligned}$$

Since every feasible point is also a regular point (the gradient of the only constraint function is never the zero vector) the Lagrange conditions are necessary conditons for an optimal solution. Some calculation give that the only solution (with $x_1 > 0$ and $x_2 > 0$) is
$x_1 = \dfrac{a_1}{\sqrt{2}}$, $x_2 = \dfrac{a_2}{\sqrt{2}}$, $u = 2a_1 a_2$, with area $4x_1 x_2 = 2a_1 a_2$.

5.(a)

The Lagrange function for the problem is given by

$$L(\mathbf{x},y)=f(\mathbf{x})+y\,g(\mathbf{x})=\sum_{j=1}^{n}\frac{c_j}{x_j}+y\,(\sum_{j=1}^{n}x_j-V)=-yV+\sum_{j=1}^{n}(\frac{c_j}{x_j}+yx_j).$$

The Lagrange relaxed problem PR_y is defined, for a given $y\geq 0$, as the problem of minimizing $L(\mathbf{x},y)$ with respect to $\mathbf{x}\in I\!R^n$.
But this problem separates into one problem for each variable x_j, namely

$$\text{minimize } \ell_j(x_j)=\frac{c_j}{x_j}+yx_j \;\text{ subject to }\; x_j\in[\,0.1,\,1\,]. \tag{0.1}$$

We have that $\ell_j'(x_j)=-\dfrac{c_j}{x_j^2}+y$ and $\ell_j''(x_j)=\dfrac{2c_j}{x_j^3}>0$, which implies that $\ell_j(x_j)$ is strictly convex on the interval $[\,0.1,\,1\,]$.

If $y=0$ then $\ell_j(x_j)$ is strictly decreasing, and then $x_j=1$ is the unique optimal solution to the subproblem (0.1).

If $y>0$ the unique solution to $\ell_j'(x_j)=0$ is $x_j=\sqrt{\dfrac{c_j}{y}}$ which belongs to the interval $[\,0.1,\,1\,]$ if and only if $c_j\leq y\leq 100c_j$.

From this, we conclude that the unique optimal solution $\tilde{x}_j(y)$ to the subproblem (0.1) is as follows:

$$\tilde{x}_j(y)=\begin{cases}1 & \text{if } \; 0\leq y\leq c_j,\\ \sqrt{\dfrac{c_j}{y}} & \text{if } \; c_j\leq y\leq 100c_j,\\ 0.1 & \text{if } \; y\geq 100c_j,\end{cases}$$

The dual objective function is then given by

$$\varphi(y)=L(\tilde{\mathbf{x}}(y),y)=-yV+\sum_{j=1}^{n}(\frac{c_j}{\tilde{x}_j(y)}+y\,\tilde{x}_j(y)).$$

The dual problem consists of maximizing $\varphi(y)$ with respect to $y\geq 0$.

5.(b)

Assume now that $n = 2$, $V = 1.5$, $c_1 = 1$ and $c_2 = 9$.

Then we get the following different expressions for $\varphi(y)$, depending on which of five intervals y belongs to:

1.) $0 \leq y \leq 1 \;\Rightarrow\; \tilde{x}_1(y) = 1,\ \tilde{x}_2(y) = 1,\ \varphi(y) = 10 + 0.5y,\ \varphi'(y) = 0.5.$

2.) $1 \leq y \leq 9 \;\Rightarrow\; \tilde{x}_1(y) = \dfrac{1}{\sqrt{y}},\ \tilde{x}_2(y) = 1,\ \varphi(y) = 9 + 2\sqrt{y} - 0.5y.\ \varphi'(y) = \dfrac{1}{\sqrt{y}} - 0.5.$

3.) $9 \leq y \leq 100 \;\Rightarrow\; \tilde{x}_1(y) = \dfrac{1}{\sqrt{y}},\ \tilde{x}_2(y) = \dfrac{3}{\sqrt{y}},\ \varphi(y) = 8\sqrt{y} - 1.5y.\ \varphi'(y) = \dfrac{4}{\sqrt{y}} - 1.5.$

4.) $100 \leq y \leq 900 \;\Rightarrow\; \tilde{x}_1(y) = 0.1,\ \tilde{x}_2(y) = \dfrac{3}{\sqrt{y}},\ \varphi(y) = 10 + 6\sqrt{y} - 1.4y.\ \varphi'(y) = \dfrac{3}{\sqrt{y}} - 1.4.$

5.) $900 \leq y \;\Rightarrow\; \tilde{x}_1(y) = 0.1,\ \tilde{x}_2(y) = 0.1,\ \varphi(y) = 100 - 1.3y.\ \varphi'(y) = -1.3.$

We note that φ and φ' are continuous, and φ' is decreasing. In particular,

$\varphi'(1) = 0.5 > 0$ and $\varphi'(9) = -1/6 < 0$. from which it follows that the optimal solution to the dual problem is to be found in the interval $1 \leq y \leq 9$, where $\varphi'(y) = \dfrac{1}{\sqrt{y}} - 0.5$.

The unique solution $\hat{y}$ to $\varphi'(y) = 0$ is then given by $\hat{y} = 4$, with $\varphi(\hat{y}) = 11$.

The corresponding primal solution is $(\hat{x}_1, \hat{x}_2) = (\tilde{x}_1(\hat{y}), \tilde{x}_2(\hat{y})) = (0.5, 1)$,
which is a feasible solution to the primal problem with $f(\hat{x}_1, \hat{x}_2) = 2 + 9 = 11 = \varphi(\hat{y})$.

Thus, $(\hat{x}_1, \hat{x}_2) = (0.5, 1)$ is a global optimal solution to the primal problem.