Exercise 6.6

For the linear programming problem

$$(P): \begin{cases} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{cases}$$

the dual is given by

$$(D): \begin{cases} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c. \end{cases}$$

We know that the following result:

<u>If</u> (1) x is a feasible solution to (P),

(2) y is a feasible solution to (D), and

(3) $c^T x = b^T y$,

<u>then</u> x is an optimal solution to (P) and

y is an optimal solution to (D).

We now check (1), (2), (3):

(1) We have

$$Ax = \begin{bmatrix} 3 & 2 & 1 & 3 & 3 & 2 \\ 2 & 4 & 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 10 \end{bmatrix} = b, \text{ and}$$

$$x = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \geqslant 0.$$

So x is a feasible solution to (P).

(2) We have

$$A^T y = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 7/4 \\ 5/2 \\ 7/4 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 3 \\ 2 \\ 2 \\ 3 \\ 2 \end{bmatrix} = c.$$

So y is a feasible solution to (D).

L

(3) Finally, we have

$$c^T x = \begin{bmatrix} 2 & 3 & 2 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 6 + 6 + 2 = 14,$$

while

$$b^T y = \begin{bmatrix} 14 & 16 & 10 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix} = \frac{7}{2} + 8 + \frac{5}{2} = 14.$$

So $c^T x = b^T y$.

Hence x is optimal for (P) and y is optimal for (D)

Exercise 6.7.

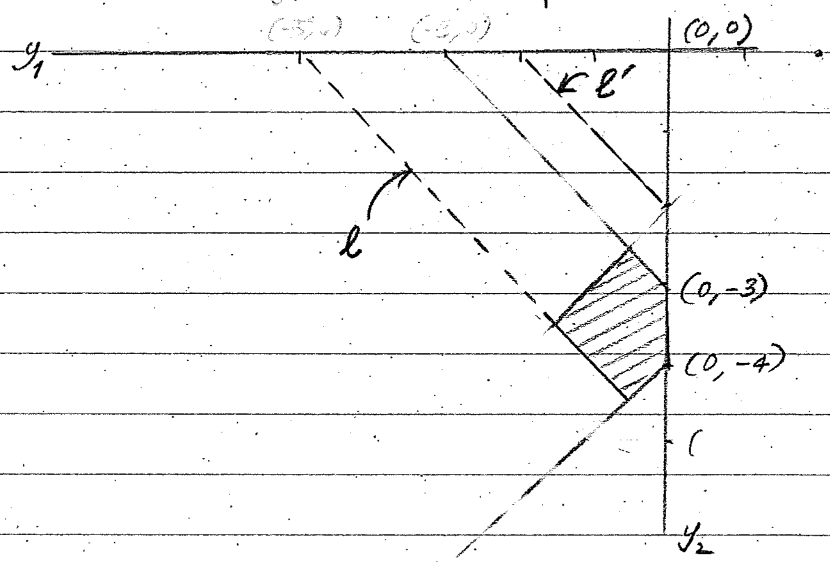
The dual problem to the primal problem (P): $\begin{cases} \text{minimize } c^T x \\ \text{subject to } Ax = b \\ x \geq 0 \end{cases}$

is given by (D): $\begin{cases} \text{maximize } b^T y \\ \text{subject to } A^T y \leq c. \end{cases}$

In our case, we obtain:

$$(D): \begin{cases} \text{maximize} & 8y_1 + 4y_2 \\ \text{subject to} & y_1 + y_2 \leq -3, \\ & y_1 - y_2 \leq 4, \\ & -y_1 + y_2 \leq -2, \\ & -y_1 - y_2 \leq 5, \\ & y_1 \leq 0, \\ & y_2 \leq 0 \end{cases}$$

The feasible set is depicted below:



When the c is changed, the constraint $-y_1 - y_2 \leq 5$ is replaced by $-y_1 - y_2 \leq 2$.
But then line ℓ moves to line ℓ' and then the feasible set for (D) is empty.
This is expected, since in Exercise 5.4, we had seen that (with the changed c), (P) has no optimal solution, but it has feasible solutions.
By Theorem 6.3 (the duality theorem), that $\mathcal{F}_D = \emptyset$ is then expected.

Exercise 6.8

Now the new primal problem is

$$
(D): \begin{cases} \text{maximize} & b_1^T y_1 + b_2^T y_2 \\ \text{subject to} & A_{11}^T y_1 + A_{21}^T y_2 \leq c_1, \\ & A_{12}^T y_1 + A_{22}^T y_2 = c_2, \\ & y_1 \geqslant 0, \quad y_2 \text{ is free} \end{cases}
$$

i.e.,
$$
(D): \begin{cases} \text{minimize} & (-b_1)^T y_1 + (-b_2)^T y_2 \\ \text{subject to} & (-A_{11}^T) y_1 + (-A_{21}^T) y_2 \geqslant -c_1, \\ & (A_{12}^T) y_1 + (A_{22}^T) y_2 = c_2, \\ & y_1 \geqslant 0, \quad y_2 \text{ is free.} \end{cases}
$$

The dual to (D) is thus given by

$$
(DD): \begin{cases} \text{maximize} & -c_1^T z_1 + c_2^T z_2 \\ \text{subject to} & (-A_{11}^T)^T z_1 + (A_{12}^T)^T z_2 \leq -b_1, \\ & (-A_{21}^T)^T z_1 + (A_{22}^T)^T z_2 = -b_2, \\ & z_1 \geqslant 0, \quad z_2 \text{ is free} \end{cases}
$$

i.e.,
$$
(DD): \begin{cases} \text{minimize} & c_1^T z_1 - c_2^T z_2 \\ \text{subject to} & A_{11} z_1 - A_{12} z_2 \geqslant b_1, \\ & A_{21} z_1 - A_{22} z_2 = b_2, \\ & z_1 \geqslant 0, \quad z_2 \text{ is free}, \end{cases}
$$

Replacing z_1 by x_1, and z_2 by $-x_2$, we obtain

$$
(DD): \begin{cases} \text{minimize} & c_1^T x_1 + c_2^T x_2 \\ \text{subject to} & A_{11} x_1 + A_{12} x_2 \geqslant b_1, \\ & A_{21} x_1 + A_{22} z_2 = b_2, \\ & x_1 \geqslant 0, \quad x_2 \text{ is free} \end{cases}
$$

which is the same as the original primal problem (P) (of which (D) is the dual).

Exercise 6.10.

The dual (D) of the primal problem (P) in standard form

$$(P): \begin{cases} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{cases}$$

is given by

$$(D): \begin{cases} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c, \end{cases}$$

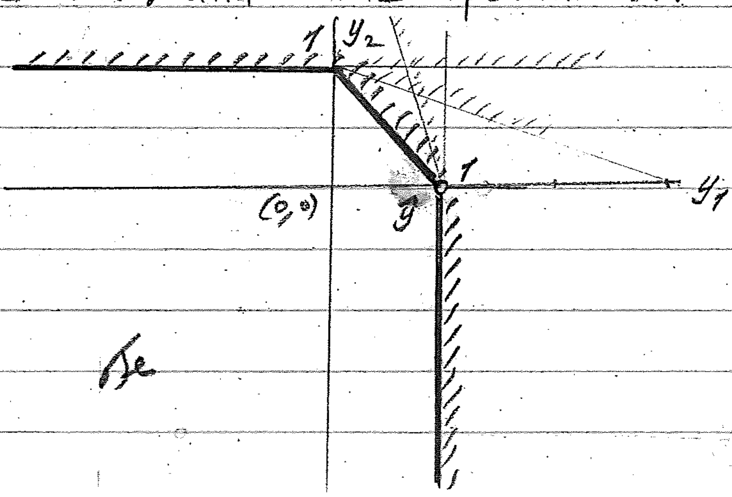
In our case, this becomes

$$(D): \begin{cases} \text{maximize} & 5y_1 + 3y_2 \\ \text{subject to} & 4y_1 \leq 4, \\ & 3y_1 + y_2 \leq 3, \\ & 2y_1 + 2y_2 \leq 2, \\ & y_1 + 3y_2 \leq 3, \\ & 4y_2 \leq 4. \end{cases}$$

We have seen that an optimal solution to the dual problem is given by the simplex multipliers vector corresponding to an optimal basic solution to the primal problem. So $\hat{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is optimal for (D).

(As a check, we can verify that $\hat{y}$ is indeed feasible for (D) and moreover, $b^T\hat{y} = 5.1 + 3.0 = 5 = c^T\hat{x}$.)

The feasible set and the optimal solution $\hat{y}$ are shown below:



Exercise 6.11

For the primal problem (P) in the standard form

$$(P): \begin{cases} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{cases}$$

the dual problem (D) is given by

$$(D): \begin{cases} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c. \end{cases}$$

In our case (see the solution to Exercise 5.7),

$$c = [1 \quad 5 \quad 2 \quad 0 \quad 0 \quad 0]^T,$$

$$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right],$$

$$b = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

So the dual problem is given by

$$(D): \begin{cases} \text{maximize} & 2y_1 + 2y_2 + 2y_3 \\ \text{subject to} & y_1 + y_2 \leq 1, \\ & y_1 + y_3 \leq 5, \\ & y_2 + y_3 \leq 2, \\ & -y_3 \leq 0, \\ & -y_2 \leq 0, \\ & -y_1 \leq 0. \end{cases}$$

We have seen that the simplex multiplier vector y corresponding to the optimal basic solution to the primal problem (P) is an optimal solution to the dual problem (D). In the solution to Exercise 5.7, we had seen that $x_1 = 2$, $x_2 = 0$, $x_3 = 2$, $x_4 = 0$, $x_5 = 2$, $x_6 = 0$ is optimal for (P) and the corresponding simplex multipliers vector is $y = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Thus this y is optimal for (D).

The optimal value for (D) is $2(y_1 + y_2 + y_3) = 2(1+0+2) = 6$, which is the same as that for (P).

Exercise 6.12.

For a primal problem (P) given by

$$(P): \begin{cases} \text{maximize} & q^T x \\ \text{subject to} & Px \leq b, \\ & x \geq 0, \end{cases}$$

the dual problem (D) is given by

$$(D): \begin{cases} \text{minimize} & b^T y \\ \text{subject to} & P^T y \geq q, \\ & y \geq 0. \end{cases}$$

In our case, this becomes

$$(D): \begin{cases} \text{minimize} & y_1 + y_2 \\ \text{subject to} & y_1 + y_2 \geq 1 \\ & -y_1 + y_2 \geq 1 \\ & y_1 - y_2 \geq 2 \\ & y_1 \geq 0 \\ & y_2 \geq 0 \end{cases}$$

But the feasible set $\mathcal{F}_D$ of (D) is empty, since there is no pair (y_1, y_2) satisfying

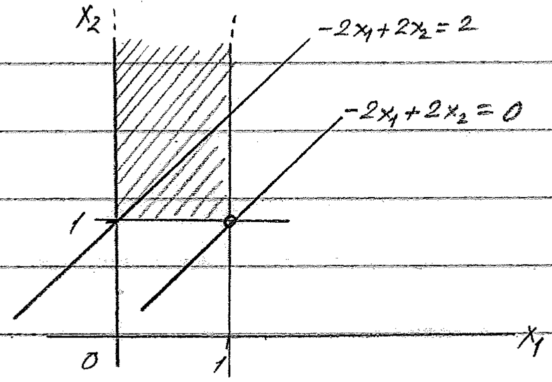
$$\begin{cases} -y_1 + y_2 \geq 1 & \text{and} \\ -(-y_1 + y_2) \geq 2 \end{cases}$$

i.e., $-2 \geq -y_1 + y_2 \geq 1$

(for otherwise $-2 \geq 1$!). So the dual problem (D) lacks feasible solutions. This is expected, since the primal problem (P) had feasible solutions, but lacked an optimal solution (see the solution to Exercise 5.8).

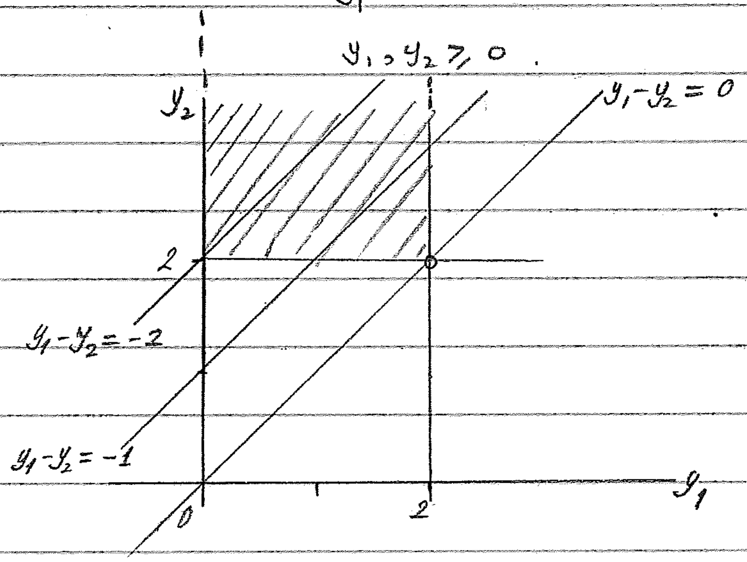
Exercise 6.13

(1) (P) : $\begin{cases} \text{minimize} & -2x_1 + 2x_2 \\ \text{subject to} & x_2 \geq 1 \\ & -x_1 \geq -1 \quad (\text{i.e., } x_1 \leq 1) \\ & x_1, x_2 \geq 0 \end{cases}$



Optimal solution for (P): $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; optimal value of (P) = 0

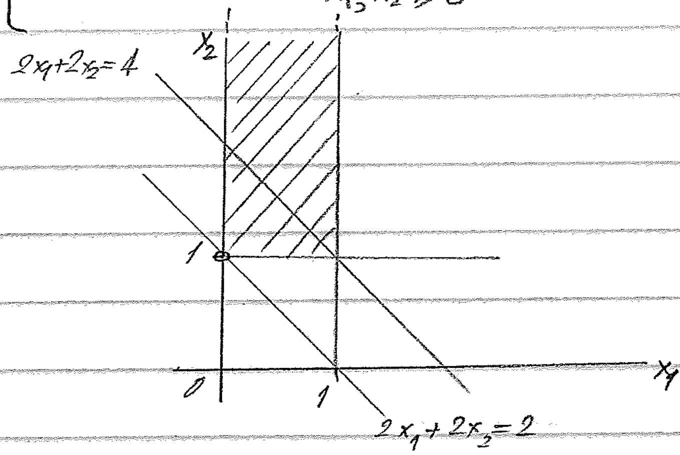
(D) : $\begin{cases} \text{maximize} & y_1 - y_2 \\ \text{subject to} & -y_2 \leq -2 \quad (\text{i.e., } y_2 \geq 2) \\ & y_1 \leq 2 \\ & y_1, y_2 \geq 0 . \end{cases}$



Optimal solution for (D) : $y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$; optimal value of (D) = 0.

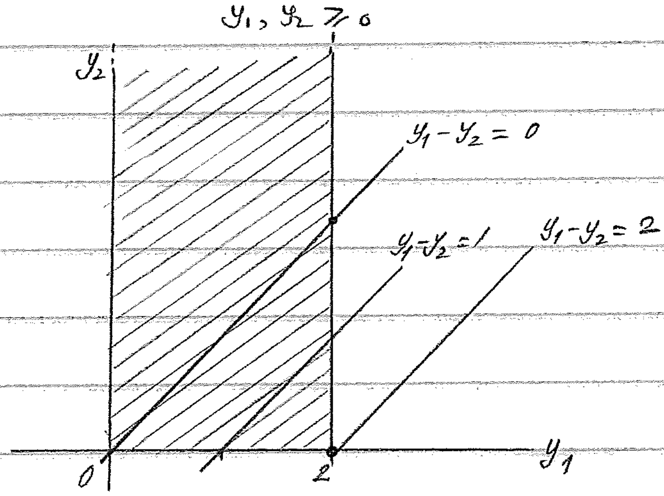
So we see that the optimal values of (P) and (D) are identical in this case (both equal to 0).

(2) (P) : $\begin{cases} \text{minimize} & +2x_1 + 2x_2 \\ \text{subject to} & x_2 \geq 1 \\ & x_1 \leq 1 \\ & x_1, x_2 \geq 0 \end{cases}$



Optimal solution for (P): $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; optimal value of (P) = 2 .

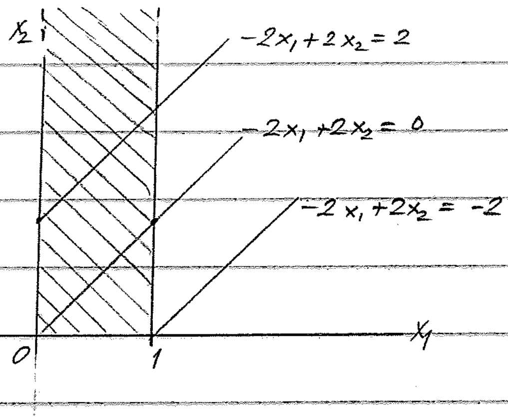
(D) : $\begin{cases} \text{maximize} & y_1 - y_2 \\ \text{subject to} & -y_2 \leq 2 \quad (\text{i.e., } y_2 \geq -2) \\ & y_1 \leq 2 \\ & y_1, y_2 \geq 0 \end{cases}$



Optimal solution for (D): $y = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$; optimal value of (D) = 2 .

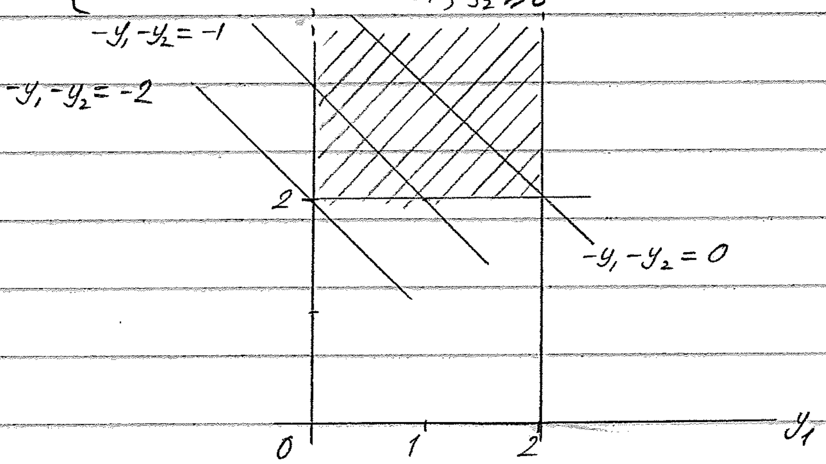
So we see that the optimal values of (P) and (D) are identical in this case (both equal to 0).

(3) (P): $\begin{cases} \text{minimize} & -2x_1 + 2x_2 \\ \text{subject to} & x_2 \geqslant -1 \\ & -x_1 \geqslant -1 \quad (\text{i.e., } x_1 \leq 1) \\ & x_1, x_2 \geqslant 0 \end{cases}$



Optimal solution for (P): $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; optimal value of (P) = -2.

(D): $\begin{cases} \text{maximize} & -y_1 - y_2 \\ \text{subject to} & y_2 \geqslant 2 \\ & y_1 \leq 2 \\ & y_1, y_2 \geqslant 0 \end{cases}$



Optimal solution for (D): $y = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$; optimal value of (D) = -2.

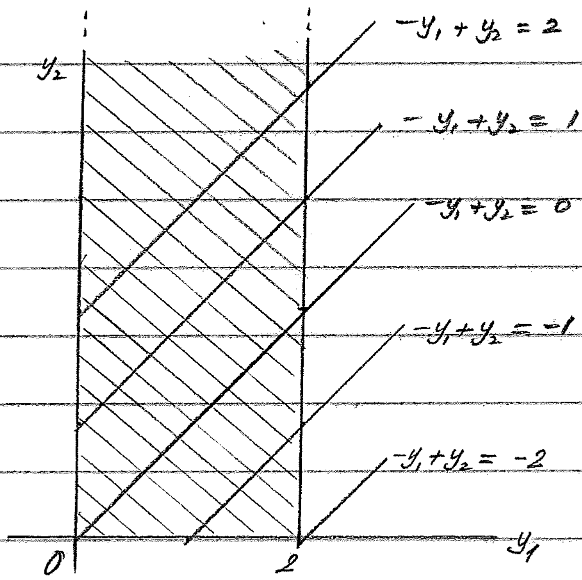
So we see that the optimal values of (P) and (D) are identical in this case (both equal to -2).

(4) (P): $\begin{cases} \text{minimize} & 2x_1 + 2x_2 \\ \text{subject to} & x_2 \geq -1 \\ & -x_1 \geq 1 \quad (\text{i.e., } x_1 \leq -1) \\ & x_1 \geq 0,\ x_2 \geq 0 \end{cases}$

The feasible set is empty.

So the optimal value is $\inf \emptyset = +\infty$.

(D): $\begin{cases} \text{maximize} & -y_1 + y_2 \\ \text{subject to} & -y_2 \leq 2 \quad (\text{i.e., } y_2 \geq -2) \\ & y_1 \leq 2 \\ & y_1, y_2 \geq 0 \end{cases}$



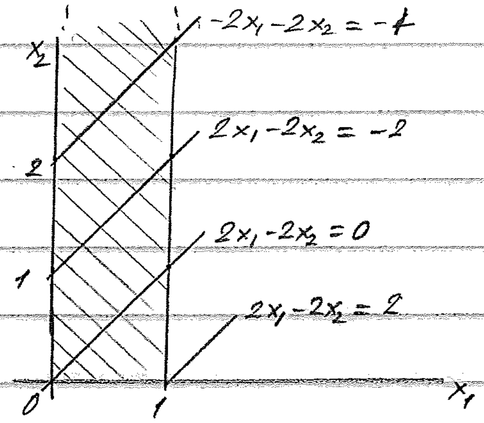
There is no optimal solution,

and the optimal value of (D) is $+\infty$

(since, for example, the points $y_n = \begin{bmatrix} 0 \\ n \end{bmatrix}$, $n \in \mathbb{N}$, are all feasible, and $b^T y_n = -0 + n = n \nearrow +\infty$ as $n \to +\infty$).

So we see that the optimal values of (P) and (D) are identical in this case (both equal to $+\infty$).

(5) (P): $\begin{cases} \text{minimize} & 2x_1 - 2x_2 \\ \text{subject to} & x_2 \geq -1 \\ & -x_1 \geq -1 \quad (\text{i.e., } x_1 \leq 1) \\ & x_1, x_2 \geq 0 \end{cases}$



There is no optimal solution, and the optimal value of (P) is $-\infty$. (For example, $x_n = \begin{bmatrix} 0 \\ n \end{bmatrix}$, $n \in \mathbb{N}$, are all feasible, and $c^T x_n = 2 \cdot 0 - 2 \cdot n = -2n \searrow -\infty$ as $n \nearrow +\infty$.)

(D): $\begin{cases} \text{maximize} & -y_1 - y_2 \\ \text{subject to} & -y_2 \leq 2 \quad \text{i.e., } y_2 \geq -2 \\ & y_1 \leq -2 \\ & y_1 \geq 0, \ y_2 \geq 0. \end{cases}$

The feasible set is empty. Hence the optimal value of (D) is $\sup \emptyset = -\infty$.

So we see that the optimal values of (P) and (D) are identical in this case (both equal to $-\infty$).

Exercise 6.14

The primal problem (LP) is in standard form:

$$(LP): \begin{cases} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geqslant 0. \end{cases}$$

Its dual is given by

$$\begin{aligned} &\text{maximize} && b^T y \\ &\text{subject to} && A^T y \leq c. \end{aligned}$$

In our case, this becomes

$$(D): \begin{cases} \text{maximize} & y_1 + 3y_2 + 5y_3 \\ \text{subject to} & y_1 + 2y_2 + 2y_3 \leq 2, \\ & y_1 + 2y_2 \leq 1, \\ & 2y_2 + 2y_3 \leq 1, \\ & y_1 + y_2 + y_3 \leq 1, \\ & 2y_3 \leq 2, \\ & 2y_2 + y_3 \leq 1. \end{cases}$$

In the solution to Exercise 5-9, we had found an optimal basic feasible solution using the simplex method, and the corresponding simplex multipliers vector was given by $y = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$. This is an optimal solution for (D)

The optimal value of (D) is $\frac{1}{2} + 3\left(-\frac{1}{2}\right) + 5(1) = 4$, while the optimal value of (LP) was

$$2 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 + 1 \cdot 0 = 4,$$ and as expected, these are the same.

Exercise 6.15

The dual problem to the problem (CLP) from Exercise 5.10 is given by

$$\begin{cases} \text{maximize} & b^T z \\ \text{subject to} & \begin{bmatrix} A^T \\ I \end{bmatrix} z \leq \begin{bmatrix} 0 \\ \hline 1 \\ 1 \end{bmatrix} \end{cases}$$

i.e.,

$$(D): \begin{cases} \text{maximize} & b^T z \\ \text{subject to} & A^T z \leq 0 \\ & z_1 \leq 1 \\ & z_2 \leq 1 \\ & z_3 \leq 1 \end{cases}$$

Since (CLP) has optimal value 1 (see the solution to Exercise 5.10), (D) has optimal value 1 as well. So every optimal solution z to (D) satisfies

$$\left.\begin{aligned} & b^T z = 1 > 0 \\ \text{and} \quad & A^T z \leq 0 \quad \text{i.e.,} \quad a_j^T z \leq 0 \text{ for all } j. \end{aligned}\right\} (*)$$

An optimal solution to the dual problem is given by the vector of multipliers corresponding to the optimal basic solution of the primal problem (CLP). Thus from the solution to Exercise 5.10, we see that $z = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an optimal solution for (D). This serves as a particular instance of a vector z satisfying $(*)$.

Exercise 6.16

The dual problem is

$$(D): \begin{cases} \text{maximize} & -c^T y \\ \text{subject to} & A^T y \leq c, \\ & y \geq 0. \end{cases}$$

But since $A^T = -A$, the dual problem becomes

$$(D): \begin{cases} \text{maximize} & -c^T y \\ \text{subject to} & -Ay \leq c, \\ & y \geq 0. \end{cases}$$

This can be rewritten as

$$(D): \begin{cases} \text{minimize} & c^T y \\ \text{subject to} & Ay \geq -c \\ & y \geq 0, \end{cases}$$

which is the same as the primal problem. Since $\mathcal{F}_P \neq \emptyset$ (given), from the above we obtain $\mathcal{F}_D = \mathcal{F}_P \neq \emptyset$ as well. So by the Duality Theorem, both the primal problem (P) as well as the dual problem (D) have optimal solutions. Also since (P) = (D), the set of their optimal solutions is the same. Hence if $\hat{x}$ is optimal for (P), then it is optimal for (D). From the Duality Theorem, we then obtain that $c^T \hat{x} = -c^T \hat{x}$, and so $c^T \hat{x} = 0$. So the optimal value of (P) is zero.